

Quantitative Languages^{*}

Krishnendu Chatterjee¹, Laurent Doyen², and Thomas A. Henzinger²

¹ University of California, Santa Cruz

² EPFL, Lausanne, Switzerland

Abstract. Quantitative generalizations of classical languages, which assign to each word a real number instead of a boolean value, have applications in modeling resource-constrained computation. We use weighted automata (finite automata with transition weights) to define several natural classes of quantitative languages over finite and infinite words; in particular, the real value of an infinite run is computed as the maximum, limsup, liminf, limit average, or discounted sum of the transition weights. We define the classical decision problems of automata theory (emptiness, universality, language inclusion, and language equivalence) in the quantitative setting and study their computational complexity. As the decidability of language inclusion remains open for some classes of weighted automata, we introduce a notion of quantitative simulation that is decidable and implies language inclusion. We also give a complete characterization of the expressive power of the various classes of weighted automata. In particular, we show that most classes of weighted automata cannot be determinized.

1 Introduction

The automata-theoretic approach to verification is boolean. To check that a system satisfies a specification, we construct a finite automaton A to model the system and a finite (usually nondeterministic) automaton B for the specification. The language $L(A)$ of A contains all behaviors of the system, and $L(B)$ contains all behaviors allowed by the specification. The language of an automaton A can be seen as a boolean function L_A that assigns 1 (or true) to words in $L(A)$, and 0 (or false) to words not in $L(A)$. The verification problem “does the system satisfy the specification?” is then formalized as the language-inclusion problem “is $L(A) \subseteq L(B)$?”, or equivalently, “is $L_A(w) \leq L_B(w)$ for all words w ?”. We present a natural generalization of this framework: a *quantitative language* L is a function that assigns a real-numbered value $L(w)$ to each (finite or infinite) word w . With quantitative languages, systems and specifications can be formalized more accurately. For example, a system may use a varying amount of some resource (e.g., memory consumption, or power consumption) depending on its behavior, and a specification may assign a maximal amount of available resource to each behavior, or fix the long-run average available use of the resource. The quantitative language-inclusion problem “is $L_A(w) \leq L_B(w)$ for all words w ?” can then be used

^{*} A preliminary version of this paper appeared in the Proceedings of the 17th International Conference on Computer Science Logic (CSL 2008). This research was supported in part by the NSF grants CCR-0132780, CNS-0720884, and CCR-0225610, by the Swiss National Science Foundation, and by the European COMBEST project.

to check, say, if for each behavior, the peak power used by the system lies below the bound given by the specification; or if for each behavior, the long-run average response time of the system lies below the specified average response requirement.

In the boolean automaton setting, the value of a word w in $L(A)$ is the maximal value of a run of A over w (if A is nondeterministic, then there may be many runs of A over w), and the value of a run is a function that depends on the class of automata: for automata over finite words, the value of a run is true if the last state of the run is accepting; for Büchi automata, the value is true if an accepting state is visited infinitely often; etc. To define quantitative languages, we use automata with weights on transitions. We again set the value of a word w as the maximal value of all runs over w , and the value of a run r is a function of the (finite or infinite) sequence of weights that appear along r . We consider several functions, such as Max and Sum of weights for finite runs, and Sup, LimSup, LimInf, limit average, and discounted sum of weights for infinite runs. For example, peak power consumption can be modeled as the maximum of a sequence of weights representing power usage; energy use can be modeled as the sum; average response time as the limit average [CCH⁺05,CdHS03]. Quantitative languages have also been used to specify and verify reliability requirements: if a special symbol \perp is used to denote failure and has weight 1, while the other symbols have weight 0, one can use a limit-average automaton to specify a bound on the rate of failure in the long run [CGH⁺08]. Alternatively, the discounted sum can be used to specify that failures happening later are less important than those happening soon [dAHM03]. It should be noted that LimSup and LimInf automata generalize Büchi and coBüchi automata, respectively. Functions such as limit average (or mean payoff) and discounted sum are classical in game theory [Sha53]; they have been studied extensively in the branching-time context of games played on graphs [EM79,Con92,CdHS03,Gim06], and it is therefore natural to consider the same functions in the linear-time context of automata and languages.

We attempt a systematic study of quantitative languages defined by weighted automata. The main novelties concern quantitative languages of infinite words, and especially those that have no boolean counterparts (i.e., limit-average and discounted-sum languages). In the first part, we consider generalizations of the boolean decision problems of emptiness, universality, language inclusion, and language equivalence. The quantitative emptiness problem asks, given a weighted automaton A and a rational number ν , whether there exists a word w such that $L_A(w) \geq \nu$. This problem can be reduced to a one-player game with a quantitative objective and is therefore solvable in polynomial time. The quantitative universality problem asks whether $L_A(w) \geq \nu$ for all words w . This problem can be formulated as a two-player game (one player choosing input letters and the other player choosing successor states) with imperfect information (the first player, whose goal is to construct a word w such that $L_A(w) < \nu$, is not allowed to see the state chosen by the second player). The problem is PSPACE-complete for simple functions like Sup, LimSup, and LimInf, but we do not know if it is decidable for limit-average or discounted-sum automata (the corresponding games of imperfect information are not known to be decidable either). The same situation holds for the quantitative language-inclusion and language-equivalence problems, which ask, given two weighted automata A and B , if $L_A(w) \leq L_B(w)$ (resp. $L_A(w) = L_B(w)$) for all

words w . Therefore we introduce a notion of quantitative simulation between weighted automata, which generalizes boolean simulation relations, is decidable, and implies language inclusion. Simulation can be seen as a weaker version of the above game, where the first player has perfect information about the state of the game. In particular, we show that quantitative simulation can be decided in $\text{NP} \cap \text{coNP}$ for limit-average and discounted-sum automata.

In the second part of this paper, we present a complete characterization of the expressive power of the various classes of weighted automata, by comparing the classes of quantitative languages they can define. The complete picture relating the expressive powers of weighted automata is shown in Fig. 7 and Table 3. For instance, the results for LimSup and LimInf are analogous to the special boolean cases of Büchi and coBüchi (nondeterminism is strictly more expressive for LimSup , but not for LimInf). In the limit-average and discounted-sum cases, nondeterministic automata are strictly more expressive than their deterministic counterparts. Also, one of our results shows that nondeterministic limit-average automata are not as expressive as deterministic Büchi automata (and vice versa). It may be noted that deterministic Büchi languages are complete for the second level of the Borel hierarchy [Tho97], and deterministic limit-average languages are complete for the third level [Cha07a]; so there is a Wadge reduction [Wad84] from deterministic Büchi languages to deterministic limit-average languages. Our result shows that Wadge reductions are not captured by automata, and in particular, that the Wadge reduction from Büchi to limit-average languages is not regular.

Other researchers have considered generalizations of languages, but as far as we know, nobody has addressed the quantitative language setting presented here. The lattice automata of [KL07] map finite words to values from a finite lattice. Roughly speaking, the value of a run is the meet (greatest lower bound) of its transition weights, and the value of a word w is the join (least upper bound) of the values of all runs over w . This corresponds to Min and Inf automata in our setting, and for infinite words, the Büchi lattice automata of [KL07] are analogous to our LimSup automata. However, the other classes of weighted automata (Sum, limit-average, discounted-sum) cannot be defined using operations on finite lattices. The complexity of the emptiness and universality problems for lattice automata is given in [KL07] (and implies our results for LimSup automata), while their generalization of language inclusion differs from ours. They define the *implication value* $v(A, B)$ of two lattice automata A and B as the meet over all words w of the join of $\neg L_A(w)$ and $L_B(w)$, while we use $+$ instead of join and define $v(A, B)$ as $\min_w (L_B(w) - L_A(w))$.

In classical weighted automata [Sch61, Moh97] and semiring automata [KS86], the value of a finite word is defined using the two algebraic operations $+$ and \cdot of a semiring as the sum of the product of the transition weights of the runs over the word. In that case, quantitative languages are called *formal power series*. Over infinite words, weighted automata with discounted sum were first investigated in [DK03]. Researchers have also considered other quantitative generalizations of languages over finite words [DG07], over trees [DKR08], and using finite lattices [GC03]. However, these works do not address the quantitative decision problems, nor do they compare the relative expressive powers of weighted automata over infinite words, as we do here. In [CCH⁺05], a quan-

titative generalization of languages is defined by discrete functions (the value of a word is an integer) and the decision problems only involve the extremal value of a language, which corresponds to emptiness.

In models that use transition weights as probabilities, such as probabilistic *Rabin automata* [Paz71], one does not consider values of individual infinite runs (which would usually have a value, or measure, of 0), but only measurable sets of infinite runs (where basic open sets are defined as extensions of finite runs). Our quantitative setting is orthogonal to the probabilistic framework: we assign quantitative values (e.g., peak power consumption, average response time, failure rate) to individual infinite behaviors, not probabilities to finite behaviors.

2 Boolean and Quantitative Languages

We recall the classical automata-theoretic description of boolean languages, and introduce an automata-theoretic description of several classes of quantitative languages.

2.1 Boolean Languages

A *boolean language* over a finite alphabet Σ is either a set $L \subseteq \Sigma^*$ of finite words or a set $L \subseteq \Sigma^\omega$ of infinite words. Alternatively, we can view these sets as functions in $[\Sigma^* \rightarrow \{0, 1\}]$ and $[\Sigma^\omega \rightarrow \{0, 1\}]$, respectively.

Boolean automata. A (*finite*) *automaton* is a tuple $A = \langle Q, q_I, \Sigma, \delta \rangle$ where:

- Q is a finite set of states, and $q_I \in Q$ is the initial state;
- Σ is a finite alphabet;
- $\delta \subseteq Q \times \Sigma \times Q$ is a finite set of labelled transitions;

The automaton A is *total* if for all $q \in Q$ and $\sigma \in \Sigma$, there exists $(q, \sigma, q') \in \delta$ for at least one $q' \in Q$. The automaton A is *deterministic* if for all $q \in Q$ and $\sigma \in \Sigma$, there exists $(q, \sigma, q') \in \delta$ for exactly one $q' \in Q$. We sometimes call automata *nondeterministic* to emphasize that they are not necessarily deterministic.

A *run* of A over a finite (resp. infinite) word $w = \sigma_1\sigma_2\dots$ is a finite (resp. infinite) sequence $r = q_0\sigma_1q_1\sigma_2\dots$ of states and letters such that (i) $q_0 = q_I$, and (ii) $(q_i, \sigma_{i+1}, q_{i+1}) \in \delta$ for all $0 \leq i < |w|$. When the run r is finite, we denote by $\text{Last}(r)$ the last state in r . When r is infinite, we denote by $\text{Inf}(r)$ the set of states that occur infinitely many times in r . The prefix of length i of an infinite run r is the prefix of r that contains the first i states.

Given a set $F \subseteq Q$ of final (or accepting) states, the *finite-word language* defined by the pair $\langle A, F \rangle$ is $L_A^f = \{w \in \Sigma^* \mid \text{there exists a run } r \text{ of } A \text{ over } w \text{ such that } \text{Last}(r) \in F\}$. The *infinite-word languages* defined by $\langle A, F \rangle$ are as follows: if $\langle A, F \rangle$ is interpreted as a Büchi automaton, then $L_A^b = \{w \in \Sigma^\omega \mid \text{there exists a run } r \text{ of } A \text{ over } w \text{ such that } \text{Inf}(r) \cap F \neq \emptyset\}$, and if $\langle A, F \rangle$ is interpreted as a coBüchi automaton, then $L_A^c = \{w \in \Sigma^\omega \mid \text{there exists a run } r \text{ of } A \text{ over } w \text{ such that } \text{Inf}(r) \subseteq F\}$.

Boolean decision problems. We recall the classical decision problems for automata, namely, emptiness, universality, language inclusion and language equivalence. Given a finite automaton A , the *boolean emptiness problem* asks whether $L_A^f = \emptyset$ (or $L_A^b = \emptyset$, or $L_A^c = \emptyset$), and the *boolean universality problem* asks whether $L_A^f = \Sigma^*$ (or $L_A^b = \Sigma^\omega$, or $L_A^c = \Sigma^\omega$). Given two finite automata A and B , the *boolean language-inclusion problem* asks whether $L_A \subseteq L_B$, and the *boolean language-equivalence problem* asks whether $L_A = L_B$. It is well-known that for both finite- and infinite-word languages, the emptiness problem is solvable in polynomial time, while the universality, inclusion, and equivalence problems are PSPACE-complete [MS72,SVW87].

2.2 Quantitative Languages

A *quantitative language* L over a finite alphabet Σ is either a mapping $L : \Sigma^+ \rightarrow \mathbb{R}$ or a mapping $L : \Sigma^\omega \rightarrow \mathbb{R}$, where \mathbb{R} is the set of real numbers.

Weighted automata. A *weighted automaton* is a tuple $A = \langle Q, q_I, \Sigma, \delta, \gamma \rangle$ where:

- $\langle Q, q_I, \Sigma, \delta \rangle$ is a total finite automaton, and
- $\gamma : \delta \rightarrow \mathbb{Q}$ is a *weight* function, where \mathbb{Q} is the set of rational numbers.

Given a finite (resp. infinite) run $r = q_0\sigma_1q_1\sigma_2\ldots$ of A over a finite (resp. infinite) word $w = \sigma_1\sigma_2\ldots$, let $\gamma(r) = v_0v_1\ldots$ be the sequence of weights that occur in r , where $v_i = \gamma(q_i, \sigma_{i+1}, q_{i+1})$ for all $0 \leq i < |w|$.

Given a *value function* $\text{Val} : \mathbb{Q}^+ \rightarrow \mathbb{R}$ (resp. $\text{Val} : \mathbb{Q}^\omega \rightarrow \mathbb{R}$), the *Val-automaton* A defines the quantitative language L_A such that for all words $w \in \Sigma^+$ (resp. $w \in \Sigma^\omega$), we have $L_A(w) = \sup\{\text{Val}(\gamma(r)) \mid r \text{ is a run of } A \text{ over } w\}$.

In sequel we denote by n the number of states and by m the number of transitions of a given automaton. We assume that rational numbers that are given as pairs of integers, encoded in binary. All time bounds we give in this paper assume that the size of the largest integer in the input is a constant p . Without this assumption, most complexity results would involve a factor p^2 , as we use only addition, multiplication, and comparison of rational numbers, which are quadratic operations.

Quantitative decision problems. We now present quantitative generalizations of the classical decision problems for automata. Given two quantitative languages L_1 and L_2 over Σ , we write $L_1 \sqsubseteq L_2$ if $L_1(w) \leq L_2(w)$ for all words $w \in \Sigma^+$ (resp. $w \in \Sigma^\omega$). Given a weighted automaton A and a rational number $\nu \in \mathbb{Q}$, the *quantitative emptiness problem* asks whether there exists a word $w \in \Sigma^+$ (resp. $w \in \Sigma^\omega$) such that $L_A(w) \geq \nu$, and the *quantitative universality problem* asks whether $L_A(w) \geq \nu$ for all words $w \in \Sigma^+$ (resp. $w \in \Sigma^\omega$). Given two weighted automata A and B , the *quantitative language-inclusion problem* asks whether $L_A \sqsubseteq L_B$, and the *quantitative language-equivalence problem* asks whether $L_A = L_B$, that is, whether $L_A(w) = L_B(w)$ for all $w \in \Sigma^+$ (resp. $w \in \Sigma^\omega$). All results that we present in this paper also hold for the decision problems defined above with inequalities replaced by strict inequalities.

Our purpose is the study of the quantitative decision problems for infinite-word languages and the expressive power of weighted automata that define infinite-word languages. We start with a brief overview of the corresponding results for finite-word languages, most of which follow from classical results in automata theory.

Finite words. For finite words, we consider the value functions Last, Max, and Sum such that for all finite sequences $v = v_1 \dots v_n$ of rational numbers,

$$\text{Last}(v) = v_n, \quad \text{Max}(v) = \max\{v_i \mid 1 \leq i \leq n\}, \quad \text{Sum}(v) = \sum_{i=1}^n v_i.$$

Note that Last generalizes the classical boolean acceptance condition for finite words. One could also consider the value function $\text{Min} = \min\{v_i \mid 1 \leq i \leq n\}$, which roughly corresponds to lattice automata [KL07].

	\exists	\forall	\subseteq	$=$
L^f	PTIME	PSPACE	PSPACE	PSPACE
L^b	PTIME	PSPACE	PSPACE	PSPACE
L^c	PTIME	PSPACE	PSPACE	PSPACE

Table 1. Complexity's upper bound for boolean decision problems (\exists) emptiness, (\forall) universality, (\subseteq) inclusion, and ($=$) equivalence.

Theorem 1. *The quantitative emptiness problem can be solved in linear time for Last and Max-automata, and in quadratic time for Sum-automata. The quantitative language-inclusion problem is PSPACE-complete for Last- and Max-automata.*

The complexity of the quantitative emptiness problem for Last and Max-automata is obtained by reduction to reachability in graphs, and for Sum-automata, by reduction to reachability of a cycle with positive value (given a Sum-automaton A with n states, there exists a finite word w such that $L_A(w) \geq \nu$ if and only if either there exists a run in A of length at most n and value at least ν , or there exists a reachable cycle with positive value). This can be checked by an iterative algorithm in quadratic time. The proof of PSPACE-completeness of the quantitative language-inclusion problem for Last- and Max-automata can be found in the appendix. The quantitative language-inclusion problem is undecidable for Sum-automata [Kro92]. However, the quantitative language-inclusion problem for deterministic Sum-automata can be solved in polynomial time using a product construction. This naturally raises the question of the power of non-determinism, which we address through translations between weighted automata.

Expressiveness. A class \mathcal{C} of weighted automata *can be reduced* to a class \mathcal{C}' of weighted automata if for every $A \in \mathcal{C}$ there exists $A' \in \mathcal{C}'$ such that $L_A = L_{A'}$. In particular, a class of weighted automata *can be determinized* if it can be reduced to its deterministic counterpart. All reductions that we present in this paper are constructive: when \mathcal{C} can be reduced to \mathcal{C}' , we always construct the automaton $A' \in \mathcal{C}'$ that defines the same quantitative language as the given automaton $A \in \mathcal{C}$. We say that the *cost* of a reduction is $O(f(n, m))$ if for all automata $A \in \mathcal{C}$ with n states and m transitions, the constructed automaton $A' \in \mathcal{C}'$ has at most $O(f(n, m))$ many states. For all reductions we present, the size of the largest transition weight in A' is linear in the size p

of the largest weight in A (however, the time needed to compute these weights may be quadratic in p).

It is easy to show that Last- and Max-automata can be determinized using a subset construction, while Sum-automata cannot be determinized. Results about determinizable sub-classes of Sum-automata can be found in [Moh97,KM05].

Theorem 2 (see also [Moh97]). *Last- and Max-automata can be determinized in $O(2^n)$ time; Sum-automata cannot be determinized. Deterministic Max-automata can be reduced to deterministic Last-automata in $O(n \cdot m)$ time; deterministic Last-automata can be reduced to deterministic Sum-automata in $O(n \cdot m)$ time. Deterministic Sum-automata cannot be reduced to Last-automata; deterministic Last-automata cannot be reduced to Max-automata.*

Proof. The proof that Last- and Max-automata can be determinized uses a variant of the subset construction, similar to what is done in the proof of Theorem 1.

To show that Sum-automata cannot be determinized, consider the language L_N over $\Sigma = \{a, b\}$ that assigns to each finite word $w \in \Sigma^+$ the number $\max\{L_a(w), L_b(w)\}$ where $L_\sigma(w)$ is the number of occurrences of σ in w (for $\sigma = a, b$). Clearly L is definable by a nondeterministic Sum-automaton. To obtain a contradiction, assume that L is defined by a deterministic Sum-automaton A with n states. Consider the word $w = a^n$ and let $r = q_0 q_1 \dots q_n$ be the unique run of A over w . There must exist $0 \leq i < j \leq n$ such that $q_i = q_j$, and thus $L_A(a^i) = L_A(a^j)$ since $L_A(a^i b^n) = L_A(a^j b^n) = n$. This is a contradiction since $L_N(a^i) = i \neq j = L_N(a^j)$.

We reduce Sup-automata to Last-automata as follows. Given a deterministic Sup-automaton $A = \langle Q, q_I, \Sigma, \delta, \gamma \rangle$, we construct the deterministic Last-automaton $A' = \langle Q', q'_I, \Sigma, \delta', \gamma' \rangle$ as follows:

- $Q' = Q \times V$ where V is the set of weights that appear on transitions of A ;
- $q'_I = (q_I, v_{\min})$ where v_{\min} is the minimal weight in V ;
- δ' contains all the tuples $\langle (q, v), \sigma, (q', v') \rangle$ such that:
 - $(q, \sigma, q') \in \delta$, and
 - $v' = v$ if $\gamma(q, \sigma, q') \leq v$, and $v' = \gamma(q, \sigma, q')$ otherwise;
- $\gamma'(\langle (q, v), \sigma, (q', v') \rangle) = v'$ for all $\langle (q, v), \sigma, (q', v') \rangle \in \delta'$.

It is easy to see that the Last-automaton A' defines the same language as A . To show that the class of Last-automata is not reducible to the class of Sup-automata, observe that the languages L defined by Sup-automata are such that $L(w_1) \leq L(w_1.w_2)$ for all $w_1, w_2 \in \Sigma^+$. It is easy to construct a Last-automaton that violates this property (consider a transition with weight v , followed by a transition with weight $v' < v$).

We reduce Last-automata to Sum-automata as follows. Given a deterministic Last-automaton $A = \langle Q, q_I, \Sigma, \delta, \gamma \rangle$, we construct the deterministic Sum-automaton $A' = \langle Q', (q'_I, 0), \Sigma, \delta', \gamma' \rangle$ as follows:

- $Q' = Q \times (V \cup \{0\})$ where V is the set of weights that appear on transitions of A ;
- $q'_I = (q_I, 0)$;
- δ' contains all the tuples $\langle (q, v), \sigma, (q', v') \rangle$ for each $v \in V \cup \{0\}$ such that:
 - $(q, \sigma, q') \in \delta$ and $v' = \gamma(q, \sigma, q')$;

$$- \gamma'(\langle (q, v), \sigma, (q', v') \rangle) = v' - v \text{ for all } \langle (q, v), \sigma, (q', v') \rangle \in \delta'.$$

It is easy to see that the Sum-automaton A' defines the same language as A . To show that the class of Sum-automata is not reducible to the class of Last-automata, observe that a Sum-automaton can define a language with infinitely many different values (e.g. counting the number of a 's in words), while Last-automata can only assign finitely many different values. ■

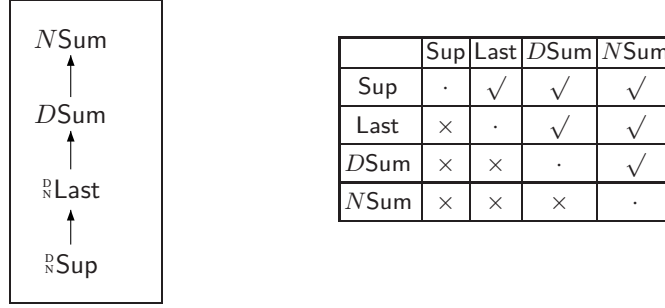


Fig. 1. Reducibility relation. \mathcal{C} is reducible to \mathcal{C}' if $\mathcal{C} \rightarrow^* \mathcal{C}'$. Classes that are not connected by an arrow are incomparable.

Infinite words. For infinite words, we consider the following classical value functions from \mathbb{Q}^ω to \mathbb{R} . Given an infinite sequence $v = v_0 v_1 \dots$ of rational numbers, define

- $\text{Sup}(v) = \sup\{v_n \mid n \geq 0\}$;
- $\text{LimSup}(v) = \limsup_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} \sup\{v_i \mid i \geq n\}$;
- $\text{LimInf}(v) = \liminf_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} \inf\{v_i \mid i \geq n\}$;
- $\text{LimAvg}(v) = \liminf_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} v_i$;
- given a discount factor $0 < \lambda < 1$, $\text{Disc}_\lambda(v) = \sum_{i=0}^{\infty} \lambda^i \cdot v_i$.

For decision problems, we always assume that the discount factor λ is a rational number. Note that $\text{LimAvg}(v)$ is defined using \liminf and is therefore well-defined; all results

of this paper hold also if the limit average of v is defined instead as $\limsup_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} v_i$.

One could also consider the value function $\text{Inf} = \inf\{v_n \mid n \geq 0\}$ and obtain results analogous to the Sup value function.

Notation. Classes of automata are sometimes denoted by acronyms of the form xyW where x is either N(ondeterministic) or D(eterministic), and y is one of the following: B(üchi), C(oBüchi), SUP, LS (LimSup), LI (LimInf), LA (LimAvg), or DI (Disc).

3 The Complexity of Quantitative Decision Problems

We study the complexity of the quantitative decision problems for weighted automata over infinite words.

	\exists	\forall	\subseteq	$=$
Sup	PTIME	PSPACE	PSPACE	PSPACE
LimSup	PTIME	PSPACE	PSPACE	PSPACE
LimInf	PTIME	PSPACE	PSPACE	PSPACE
LimAvg	PTIME	?	?	?
Deterministic LimAvg	PTIME	PTIME	PTIME	PTIME
Disc $_{\lambda}$	PTIME	?	?	?
Deterministic Disc $_{\lambda}$	PTIME	PTIME	PTIME	PTIME

Table 2. Complexity's upper bound for quantitative decision problems (\exists) emptiness, (\forall) universality, (\subseteq) inclusion and ($=$) equivalence.

Emptiness. The quantitative emptiness problem can be solved by reduction to the problem of finding the maximal value of an infinite path in a graph. This is decidable because pure memoryless strategies for resolving nondeterminism exist for all quantitative objectives that we consider [FV97,Kar78,And06].

Theorem 3. *The quantitative emptiness problem is solvable in $O(m+n)$ time for Sup-, LimSup-, and LimInf-automata; in $O(n \cdot m)$ time for LimAvg-automata; and in $O(n^2 \cdot m)$ time for Disc-automata.*

Proof. Given a quantitative function Val and a weighted automaton A , let

$$\text{Val}(A) = \sup\{\text{Val}(\gamma(r)) \mid w \in \Sigma^\omega, r \text{ is a run of } A \text{ over } w\}.$$

Given the quantitative function Val is Sup, LimSup, LimInf, LimAvg, or Disc $_{\lambda}$ the following assertion holds: there exists a word w^* such that $w^* = w_1 \cdot (w_2)^\omega$, for finite words w_1 and w_2 of length at most $|Q|$ (i.e., w^* is a *lasso* word), such that $\text{Val}(A) = \sup\{\text{Val}(\gamma(r)) \mid r \text{ is a run of } A \text{ over } w^*\}$. The above result follows from the fact that for transition systems (or graphs) with rewards on edges, and objectives defined by Sup, LimSup, LimInf, LimAvg, or Disc $_{\lambda}$, pure memoryless optimal strategies exist. The result for Sup, LimSup, and LimInf objectives are obtained by extending the result for reachability, Büchi and coBüchi objectives, respectively, and the result for LimAvg and Disc $_{\lambda}$ can be obtained as a special case of the result known for Markov

decision processes [FV97]. It follows that the answer to the emptiness question is “Yes” iff $\text{Val}(A) \geq \nu$. Given a weighted automaton A , let n and m denote the size of Q and δ , respectively. The algorithms for computation of $\text{Val}(A)$ is as follows:

- If Val is Sup , then $\text{Val}(A)$ can be computed in $O(m + n)$ time, by classical reachability of weights greater than (or equal to) ν ;
- If Val is LimSup or LimInf , then $\text{Val}(A)$ can be computed in $O(m + m)$ time, by the same algorithm as for Büchi and coBüchi (computing the maximal strongly connected components, and reachability to strongly connected components) where the “accepting” edges are those with a weight greater than (or equal to) ν ;
- If Val is LimAvg , then $\text{Val}(A)$ can be computed in time $O(nm)$ time by applying the maximum mean cycle algorithm [Kar78];
- If Val is Disc_λ , then $\text{Val}(A)$ can be computed in $O(n^2m)$ time by applying the algorithm to solve discounted payoff objectives in graphs with rewards on edges [And06].

■

Language inclusion. The following theorem relies on the analogous result for finite automata.

Theorem 4. *The quantitative language-inclusion problem is PSPACE-complete for Sup-, LimSup-, and LimInf-automata.*

To prove Theorem 4, we need the following lemmas, whose proof is given in the appendix.

Lemma 1. *Given a Sup-automaton $A = \langle Q_1, q_I^1, \Sigma, \delta_1, \gamma_1 \rangle$, we can construct in linear time a LimSup-automaton $B = \langle Q_2, q_I^2, \Sigma, \delta_2, \gamma_2 \rangle$ such that $L_A(w) = L_B(w)$ for all $w \in \Sigma^\omega$.*

Lemma 2. *Given a LimSup-automaton $A = \langle Q, q_I, \Sigma, \delta, \gamma \rangle$ and a rational number v , we can construct in linear time a finite automaton $A^{\geq v}$ with accepting states $F^{\geq v}$ such that $L_{A^{\geq v}}^b = \{w \in \Sigma^\omega \mid L_A(w) \geq v\}$.*

Proof of Theorem 4. We show that the quantitative language inclusion problem for LimSup-automata is PSPACE-complete. Let $A = \langle Q_1, q_I^1, \Sigma, \delta_1, \gamma_1 \rangle$ and $B = \langle Q_2, q_I^2, \Sigma, \delta_2, \gamma_2 \rangle$ be two LimSup-automata. Let V_1 be the (finite) set of weights that occur on some transitions in δ_1 . Clearly, we have $L_A(w) \in V_1$ for all words $w \in \Sigma^\omega$. Consider the finite automaton $B^{\geq v}$ of Lemma 2. The quantitative language inclusion problem $A \sqsubseteq B$ is equivalent to check that $L^b(A^{\geq v}) \subseteq L^b(B^{\geq v})$ for all $v \in V_1$, and thus it is in PSPACE. The hardness result is obtained by a simple reduction of the boolean language inclusion problem for finite automata which is PSPACE-complete. It suffices to assign weight 1 to the transitions entering an accepting state, and weight 0 to the other transitions.

The proof that the quantitative language inclusion problem for Sup- and LimInf-automata is PSPACE-complete is analogous (using Lemma 1). ■

We do not know if the quantitative language-inclusion problem is decidable for LimAvg- or Disc-automata. The special cases of deterministic automata are easy, using a product construction.

Theorem 5. *The quantitative language-inclusion problems $L_A \sqsubseteq L_B$ for LimAvg- and Disc-automata are decidable in polynomial time when B is deterministic.*

To prove Theorem 5, we need the following lemma.

Lemma 3. *For all sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ of real numbers, we have:*

- $\limsup_{n \rightarrow \infty} a_n + b_n \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$
- $\limsup_{n \rightarrow \infty} a_n - b_n \geq \limsup_{n \rightarrow \infty} a_n - \limsup_{n \rightarrow \infty} b_n$
- $\liminf_{n \rightarrow \infty} a_n + b_n \geq \liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n$
- $\liminf_{n \rightarrow \infty} a_n - b_n \leq \liminf_{n \rightarrow \infty} a_n - \liminf_{n \rightarrow \infty} b_n$

Proof. The first statement is well known. The second statement is obtained by substituting in the first a_n with b'_n , and b_n with $a'_n - b'_n$. The last two statements follow from the first two by the equality $\limsup_{n \rightarrow \infty} a_n = -\liminf_{n \rightarrow \infty} -a_n$. ■

Proof of Theorem 5. Given two weighted automata $A = \langle Q_1, q_1^1, \Sigma, \delta_1, \gamma_1 \rangle$ and $B = \langle Q_2, q_2^1, \Sigma, \delta_2, \gamma_2 \rangle$, we define the product weighted automaton as follows: $A \times B = \langle Q_1 \times Q_2, (q_1^1, q_2^1), \Sigma, \delta_{12}, \gamma_{12} \rangle$, where $((q_1, q_2), \sigma, (q'_1, q'_2)) \in \delta_{12}$ iff $(q_1, \sigma, q'_1) \in \delta_1$ and $(q_2, \sigma, q'_2) \in \delta_2$; and for $((q_1, q_2), \sigma, (q'_1, q'_2)) \in \delta_{12}$ we have $\gamma_{12}((q_1, q_2), \sigma, (q'_1, q'_2)) = \gamma_1((q_1, \sigma, q'_1)) - \gamma_2((q_2, \sigma, q'_2))$. The following assertion holds: if B is deterministic, then the answer to the quantitative inclusion problem is “No” iff

$$\text{Val}(A \times B) = \sup\{\text{Val}(\gamma_{12}(r_{12})) \mid w \in \Sigma^\omega, r_{12} \text{ is a run of } A \times B \text{ over } w\} > 0,$$

where Val is LimAvg or Disc $_\lambda$. We present both directions of the proof.

1. We first show that if $\text{Val}(A \times B) > 0$, then the answer to the quantitative inclusion problem is “No”. If $\text{Val}(A \times B) > 0$, then it follows from arguments similar to Theorem 3, that there exists a lasso word $w^* = w_1 \cdot (w_2)^\omega$ such that $L_{A \times B}(w^*) > 0$. Consider a run r_{12}^* of $A \times B$ over w^* such that $\text{Val}(\gamma_{12}(r_{12}^*)) > 0$. The run r_{12}^* can be decomposed as a run r_1^* of A over w^* and the unique run r_2^* of B over w^* (the run r_2^* is unique since B is deterministic). Let the sequence of numbers in $\gamma_1(r_1^*)$ and $\gamma_2(r_2^*)$ be $v_0^1, v_1^1, v_2^1, v_3^1, \dots$, and $v_0^2, v_1^2, v_2^2, v_3^2, \dots$, respectively. Then the sequence of numbers in $\gamma_{12}(r_{12}^*)$ is $v_0^1 - v_0^2, v_1^1 - v_1^2, v_2^1 - v_2^2, v_3^1 - v_3^2, \dots$.
 - If $\text{Val} = \text{Disc}_\lambda$, then we have

$$\text{Val}(\gamma_{12}(r_{12}^*)) = \sum_{i=0}^{\infty} \lambda^i \cdot (v_i^1 - v_i^2) = \sum_{i=0}^{\infty} \lambda^i \cdot v_i^1 - \sum_{i=0}^{\infty} \lambda^i \cdot v_i^2.$$

Since r_1^* is a run of A over w^* we have $L_A(w^*) \geq \sum_{i=0}^{\infty} \lambda^i \cdot v_i^1$, and since B is deterministic we have $L_B(w^*) = \sum_{i=0}^{\infty} \lambda^i \cdot v_i^2$. Since $\text{Val}(\gamma_{12}(r_{12}^*)) > 0$, it follows that $L_A(w^*) > L_B(w^*)$.

- If $\text{Val} = \text{LimAvg}$, then we have

$$\text{Val}(\gamma_{12}(r_{12}^*)) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} (v_i^1 - v_i^2) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^1 - \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^2.$$

The last inequality follows from Lemma 3. Since r_1^* is a run of A over w^* we have $L_A(w^*) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^1$. Moreover, since B is deterministic we have $L_B(w^*) = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^2$. Since $\text{Val}(\gamma_{12}(r_{12}^*)) > 0$, it follows that $L_A(w^*) > L_B(w^*)$.

- If Val is the lim sup version of LimAvg , i.e. $\text{Val}(v_0 v_1 \dots) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i$, then we have

$$\text{Val}(\gamma_{12}(r_{12}^*)) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} (v_i^1 - v_i^2) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^1 - \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^2.$$

The last inequality follows from arguments similar to Lemma 3, and observe that the last term is \liminf . Since the word w^* is a lasso word, and B is deterministic, the run r_2^* is a lasso run, and we have $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^2 = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^2 = L_B(w^*)$. Since r_1^* is a run of A over w^* we have $L_A(w^*) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^1$. Since $\text{Val}(\gamma_{12}(r_{12}^*)) > 0$, it follows that $L_A(w^*) > L_B(w^*)$.

2. We now prove the other direction.

- We first consider the case when $\text{Val} = \text{Disc}_\lambda$. If for some word w^* we have $L_A(w^*) > L_B(w^*)$, then consider a run r_1^* of A over w^* , and the unique run r_2^* of B over w^* such that $\text{Val}(\gamma_1(r_1^*)) > \text{Val}(\gamma_2(r_2^*))$. For the run r_{12}^* composed of the runs r_1^* and r_2^* we have $\text{Val}(\gamma_{12}(r_{12}^*)) = \text{Val}(\gamma_1(r_1^*)) - \text{Val}(\gamma_2(r_2^*))$ (this holds for Val being Disc_λ and follows from arguments similar to the case above). It follows that $\text{Val}(\gamma_{12}(r_{12}^*)) > 0$, and hence $\text{Val}(A \times B) > 0$.
- We now consider the case when $\text{Val} = \text{LimAvg}$. If $\text{Val}(A \times B) \leq 0$, then we show that for all words w we have $L_A(w) \leq L_B(w)$. Since $\text{Val}(A \times B) \leq 0$, it follows that for all cycles C reachable from (q_I^1, q_I^2) in $A \times B$ we have the sum of weights according to γ_{12} is at most 0; i.e., in C the sum of the weights according to γ_2 is at least the sum of the weights according to γ_1 . For a word w , let us consider a run r_1 in A and the unique run r_2 in B . Let the sequence of weights in r_1 and r_2 be $(v_i^1)_{i \geq 0}$ and $(v_i^2)_{i \geq 0}$, respectively. By the property of cycles in $A \times B$ (i.e., the sum of weights by γ_2 in any cycle is at least the sum of the weights by γ_1), it follows that for all $j \geq 0$ we have

$$\sum_{i=0}^j v_i^1 \leq \sum_{i=0}^j v_i^2 - 2 \cdot |Q_1 \times Q_2| \cdot \beta$$

where $\beta = \max_{((q_1, q_2), \sigma, (q'_1, q'_2)) \in \delta_{12}} |\gamma_1(q_1, \sigma, q'_1) - \gamma_2(q_2, \sigma, q'_2)|$. Hence we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^1 \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^2 - \liminf_{n \rightarrow \infty} \frac{2 \cdot |Q_1| \cdot |Q_2| \cdot \beta}{n} = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} v_i^2.$$

The result follows.

- We consider the case when Val is the \limsup version of LimAvg . If for some word w^* we have $L_A(w^*) > L_B(w^*)$, then consider a run r_1^* of A over w^* , and the unique run r_2^* of B over w^* such that $\text{Val}(\gamma_1(r_1^*)) > \text{Val}(\gamma_2(r_2^*))$. For the run r_{12}^* composed of the runs r_1^* and r_2^* we have

$$\text{Val}(\gamma_{12}(r_{12}^*)) \geq \text{Val}(\gamma_1(r_1^*)) - \text{Val}(\gamma_2(r_2^*)).$$

The above inequality follows from Lemma 3. It follows that $\text{Val}(\gamma_{12}(r_{12}^*)) > 0$, and hence $\text{Val}(A \times B) > 0$.

It follows from above that the answer to the quantitative inclusion problem is “No” iff $\text{Val}(A \times B) > 0$. It follows from arguments of Theorem 3 that $\text{Val}(A \times B)$ is computable in polynomial time, for Val being LimAvg or Disc_λ . The desired result follows. ■

When B is not deterministic, we make the following observation. There exist two LimAvg -automata A and B such that (i) $L_A \not\subseteq L_B$ and (ii) there exist no finite words w_1 and w_2 such that $L_A(w) > L_B(w)$ for $w = w_1 \cdot w_2^\omega$ (the word w is called a *lasso-word*). Consider the two LimAvg -automata A and B shown in Fig. 2, where B is nondeterministic. For all words $w \in \Sigma^\omega$, we have $L_A(w) = 1$. For a lasso-word of the form $w = w_1 \cdot w_2^\omega$, if in w_2 there are more b ’s than a ’s, then B chooses q'_3 from q'_1 , and else chooses q'_2 from q'_1 . Hence for all lasso-words $w = w_1 \cdot w_2^\omega$, we have $L_B(w) \geq 1$. However $L_A \not\subseteq L_B$. Consider the word w generated inductively such that w_0 is the empty word, and w_{i+1} is generated from w_i as follows: (i) first generate a long enough sequence w'_{i+1} of a ’s after w_i such that the average number of b ’s in $w_i \cdot w'_{i+1}$ falls below $\frac{1}{3}$; (ii) then generate a long enough sequence w''_{i+1} of b ’s such that the average number of a ’s in $w_i \cdot w'_{i+1} \cdot w''_{i+1}$ falls below $\frac{1}{3}$; and (iii) let $w_{i+1} = w_i \cdot w'_{i+1} \cdot w''_{i+1}$. The infinite word w is the limit of this sequence. For the word w , we have $L_B(w) = 2 \cdot \frac{1}{3} = \frac{2}{3} < 1$, and thus $L_A \not\subseteq L_B$. This observation is in contrast to the case of boolean language inclusion for, e.g., parity automata, where non-inclusion is always witnessed by a lasso-word. For the quantitative language-inclusion problem for discounted sum automata we have the following theorem.

Theorem 6. *The quantitative language-inclusion problem for Disc-automata is co-r.e.*

Proof. For discounted-sum automata A with weight function γ_1 and B with weight function γ_2 , assume that there exists a finite word $w \in \Sigma^*$ such that for some run r_1 of A over w and for all runs r_2 of B over w , we have

$$\gamma_1(r_1) + v \cdot \frac{\lambda^{|w|}}{1-\lambda} > \gamma_2(r_2) + V \cdot \frac{\lambda^{|w|}}{1-\lambda}$$

where v (resp. V) is the minimal (resp. maximal) weight in (the union of) A and B . Then, we immediately have $L_A \not\subseteq L_B$, as $L_A(w \cdot w') > L_B(w \cdot w')$ for all words $w' \in \Sigma^\omega$. We say that w is a *finite witness* of $L_A \not\subseteq L_B$. We claim that there always exists a finite witness of $L_A \not\subseteq L_B$. To see this, consider an infinite word w^∞ such that $L_A(w^\infty) = \eta_1$, $L_B(w^\infty) = \eta_2$, and $\eta_1 > \eta_2$. Let r_1 be an (infinite) run of A over w^∞

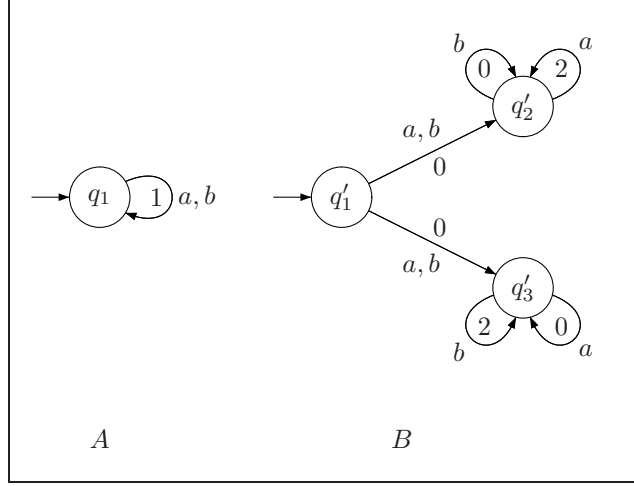


Fig. 2. Two nondeterministic limit-average automata A and B such that $L_A \not\sqsubseteq L_B$, but there is no word of the form $w = w_1 \cdot w_2^\omega$ with $L_A(w) > L_B(w)$.

whose value is η_1 . For $i > 0$, consider the prefix of w^∞ of length i . Then, for all runs r_2 of B over w^∞ , we have

$$\gamma_1(r_1^i) + V \cdot \frac{\lambda^i}{1-\lambda} \geq \eta_1 \quad \text{and} \quad \gamma_2(r_2^i) + v \cdot \frac{\lambda^i}{1-\lambda} \leq \eta_2$$

where r_1^i and r_2^i are the prefixes of length i of r_1 and r_2 , respectively. Then, a prefix of length i of w^∞ is a finite witness of $L_A \not\sqsubseteq L_B$ if

$$\eta_1 - (V - v) \cdot \frac{\lambda^i}{1-\lambda} > \eta_2 + (V - v) \cdot \frac{\lambda^i}{1-\lambda},$$

which must hold for sufficiently large values of i .

Therefore, the following procedure terminates if $L_A \not\sqsubseteq L_B$: enumerate the finite words over Σ (and all the runs of A_1 and A_2 over these words) and check the condition (3) to get a finite witness of $L_A \not\sqsubseteq L_B$. ■

Universality and language equivalence. All of the above results about language inclusion hold for quantitative universality and language equivalence also. See Theorems 17, 18, 19, and 20 in the appendix.

4 Quantitative Simulation

As the decidability of the quantitative language-inclusion problems for limit-average and discounted-sum automata remain open, we introduce a notion of *quantitative simulation* as a decidable approximation of language inclusion for weighted automata. The

quantitative language-inclusion problem can be viewed as a game of imperfect information, and we view the quantitative simulation problem as exactly the same game, but with perfect information. For quantitative objectives, perfect-information games can be solved much more efficiently than imperfect-information games, and in some cases the solution of imperfect-information games with quantitative objectives is not known. For example, perfect-information games with limit-average and discounted-sum objectives can be decided in $\text{NP} \cap \text{coNP}$, whereas the solution for such imperfect-information games is not known. Second, quantitative simulation implies quantitative language inclusion, because it is easier to win a game when information is not hidden. Hence, as in the case of finite automata, simulation can be used as a conservative and efficient approximation for language inclusion.

Language-inclusion game. Let A and B be two weighted automata with weight function γ_1 and γ_2 , respectively, for which we want to check if $L_A \sqsubseteq L_B$. The language-inclusion game is played by a *challenger* and a *simulator*, for infinitely many rounds. The goal of the simulator is to prove that $L_A \sqsubseteq L_B$, while the challenger has the opposite objective. The position of the game in the initial round is $\langle q_I^1, q_I^2 \rangle$ where q_I^1 and q_I^2 are the initial states of A and B , respectively. In each round, if the current position is $\langle q_1, q_2 \rangle$, first the challenger chooses a letter $\sigma \in \Sigma$ and a state q'_1 such that $(q_1, \sigma, q'_1) \in \delta_1$, and then the simulator chooses a state q'_2 such that $(q_2, \sigma, q'_2) \in \delta_2$. The position of the game in the next round is $\langle q'_1, q'_2 \rangle$. The outcome of the game is a pair (r_1, r_2) of runs of A and B , respectively, over the same infinite word. The simulator wins the game if $\text{Val}(\gamma_2(r_2)) \geq \text{Val}(\gamma_1(r_1))$. To make this game equivalent to the language-inclusion problem, we require that the challenger cannot observe the state of B in the position of the game.

Simulation game. The simulation game is the language-inclusion game without the restriction on the vision of the challenger, that is, the challenger is allowed to observe the full position of the game. Formally, given $A = \langle Q_1, q_I^1, \Sigma, \delta_1, \gamma_1 \rangle$ and $B = \langle Q_2, q_I^2, \Sigma, \delta_2, \gamma_2 \rangle$, a *strategy* τ for the challenger is a function from $(Q_1 \times Q_2)^+$ to $\Sigma \times Q_1$ such that for all $\pi \in (Q_1 \times Q_2)^+$, if $\tau(\pi) = (\sigma, q)$, then $(\text{Last}(\pi|_{Q_1}), \sigma, q) \in \delta_1$, where $\pi|_{Q_1}$ is the projection of π on Q_1^+ . A strategy τ for the challenger is *blind* if $\tau(\pi) = \tau(\pi')$ for all sequences $\pi, \pi' \in (Q_1 \times Q_2)^*$ such that $\pi|_{Q_1} = \pi'|_{Q_1}$. The set of *outcomes* of a challenger strategy τ is the set of pairs (r_1, r_2) of runs such that if $r_1 = q_0 \sigma_1 q_1 \sigma_2 \dots$ and $r_2 = q'_0 \sigma_1 q'_1 \sigma_2 \dots$, then $q_0 = q_I^1$, $q'_0 = q_I^2$, and for all $i \geq 0$, we have $(\sigma_{i+1}, q_{i+1}) = \tau((q_0, q'_0) \dots (q_i, q'_i))$ and $(q'_i, \sigma_{i+1}, q'_{i+1}) \in \delta_2$. A strategy τ for the challenger is *winning* if $\text{Val}(\gamma_1(r_1)) > \text{Val}(\gamma_2(r_2))$ for all outcomes (r_1, r_2) of τ .

Theorem 7. *For all value functions and weighted automata A and B , we have $L_A \sqsubseteq L_B$ iff there is no blind winning strategy for the challenger in the language-inclusion game for A and B .*

Given two weighted automata A and B , there is a *quantitative simulation* of A by B if there exists no (not necessarily blind) winning strategy for the challenger in the simulation game for A and B . We note that for the special cases of Büchi and coBüchi automata, quantitative simulation coincides with *fair simulation* [HKR97].

Corollary 1. *For all value functions and weighted automata A and B , if there is a quantitative simulation of A by B , then $L_A \sqsubseteq L_B$.*

Given two weighted automata A and B , the *quantitative simulation problem* asks if there is a quantitative simulation of A by B .

Theorem 8. *The quantitative simulation problem for Sup-automata is solvable in polynomial time. The quantitative simulation problem is in $NP \cap coNP$ for LimSup-, LimInf-, LimAvg-, and Disc-automata.*

The proof of Theorem 8 is obtained as follows. The quantitative simulation problems for LimSup- and LimInf-automata is reduced to perfect-information parity games; the quantitative simulation problem for LimAvg-automata is reduced to perfect-information limit-average games; and the quantitative simulation problem for Disc-automata is reduced to perfect-information discounted-sum games. All reductions are polynomial time, and the resulting games can all be solved in $NP \cap coNP$.

Proof of Theorem 8. First, we consider Sup-, LimSup- and LimInf-automata. Let $A = \langle Q_1, q_1^1, \Sigma, \delta_1, \gamma_1 \rangle$ and $B = \langle Q_2, q_2^1, \Sigma, \delta_2, \gamma_2 \rangle$ be two Sup- automata (or two LimSup- automata, or two LimInf-automata). Let $v_1 < v_2 < \dots < v_k$ be the weights that occur in A . We construct the game structure $\mathcal{G}(A, B) = \langle Q_{challenger}, Q_{simulator}, q_I, E, p \rangle$ where:

- $Q_{challenger} = Q_1 \times Q_2$;
- $Q_{simulator} = Q_1 \times Q_2 \times \Sigma$;
- $q_I = (q_1^1, q_2^1)$;
- $E = \{((q_1, q_2), (q'_1, q_2, \sigma)) \mid (q_1, \sigma, q'_1) \in \delta_1\} \cup \{((q_1, q_2, \sigma), (q_1, q'_2)) \mid (q_2, \sigma, q'_2) \in \delta_2\}$;
- $p : E \rightarrow \{0, 1, \dots, 2k\}$ assigns priorities to edges as follows:
 - $p((q_1, q_2), (q'_1, q_2, \sigma)) = 2i - 1$ if $\gamma_1(q_1, \sigma, q'_1) = v_i$
 - $p((q_1, q_2, \sigma), (q_1, q'_2)) = \begin{cases} 0 & \text{if } \gamma_2(q_2, \sigma, q'_2) < v_1 \\ 2i & \text{if } v_i \leq \gamma_2(q_2, \sigma, q'_2) < v_{i+1} \\ 2k & \text{if } v_k \leq \gamma_2(q_2, \sigma, q'_2) \end{cases}$

The objective of the simulator in the game for Sup-automata is that the maximal priority which occurs in a play is even, i.e. a weak-parity objective [Tho97]. The objective of the simulator in the game for LimSup-automata is that the maximal priority which is seen infinitely often is even, and in the game for LimInf-automata that the minimal priority which is seen infinitely often is odd, i.e. classical parity objectives.

In the three cases a winning strategy of the simulator in $\mathcal{G}(A, B)$ is a witness that there is no winning strategy for the challenger in the simulation game. Similarly, a winning strategy for the challenger in $\mathcal{G}(A, B)$ is a witness of a winning strategy for the challenger in the simulation game. Hence it follows that the simulator wins in $\mathcal{G}(A, B)$ if and only if B simulates A .

The $NP \cap coNP$ complexity result for LimSup- and LimInf-automata then follows from the fact that parity games can be solved in $NP \cap coNP$ [EJ91]. Since weak-parity games are solvable in linear-time [Cha08], the quantitative simulation problem for Sup-automata is solvable in polynomial time.

The simulation game for NLAW and NDIW (with a rational discount factor) can also be solved in $\text{NP} \cap \text{coNP}$. The proof constructs a game with limit-average (resp. discounted) objective. The game has the same structure (states and transitions) as in the proof of Theorem 8. Weights are assigned to transitions as follows: if it corresponds to a transition in A_1 , then it has the same weight as in A_1 , and if it corresponds to a transition in A_2 with weight v , then it has weight $-v$ for the limit-average game and $\frac{v}{\sqrt{\lambda}}$ for the discounted game (where λ is the discount factor of A_1 and A_2). Moreover, the discount factor of the discounted game is $\sqrt{\lambda}$.

Now, we consider the case of LimAvg-automata. Let $A = \langle Q_1, q_I^1, \Sigma, \delta_1, \gamma_1 \rangle$ and $B = \langle Q_2, q_I^2, \Sigma, \delta_2, \gamma_2 \rangle$ be two LimAvg-automata. We construct the quantitative perfect-information limit-average game $\mathcal{G}(A, B) = \langle Q_{\min}, Q_{\max}, q_I, E, \gamma \rangle$ where:

- $Q_{\min} = Q_1 \times Q_2$;
- $Q_{\max} = Q_1 \times Q_2 \times \Sigma$;
- $q_I = (q_I^1, q_I^2)$;
- $E = \{((q_1, q_2), (q'_1, q_2, \sigma)) \mid (q_1, \sigma, q'_1) \in \delta_1\} \cup \{((q_1, q_2, \sigma), (q_1, q'_2)) \mid (q_2, \sigma, q'_2) \in \delta_2\}$.
- γ assigns $-\gamma_1(q_1, \sigma, q'_1)$ to each $((q_1, q_2), (q'_1, q_2, \sigma)) \in E$, and $\gamma_2(q_2, \sigma, q'_2)$ to each $((q_1, q_2, \sigma), (q_1, q'_2)) \in E$.

It is easy to establish a correspondence between strategies of the challenger in the simulation game and the min-player in game $\mathcal{G}(A, B)$, and similarly, for the simulator and the max-player. The following two case analysis relates the maximal value of the perfect-information limit-average game $\mathcal{G}(A, B)$ and the simulation game. In the following analysis we use pure memoryless determinacy of perfect-information limit-average games [EM79] (i.e., existence of pure memoryless optimal strategies in such games).

1. We first show that if the maximal value that the max-player can ensure is at least 0, then B simulates A . We fix an optimal strategy τ_1^* for the max-player in $\mathcal{G}(A, B)$. Consider an arbitrary strategy for the min-player, and the resulting play π starting from q_I . Let the sequence of weights in the play be $-u_0, v_0, -u_1, v_1, -u_2, v_2, \dots$. Since the maximal value is at least 0 and τ_1^* is an optimal strategy, it follows that

$$\liminf_{n \rightarrow \infty} \frac{1}{2n} \cdot \sum_{i=0}^{n-1} (v_i - u_i) \geq 0.$$

By Lemma 3 we obtain that

$$\frac{1}{2} \left(\liminf_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} v_i - \liminf_{n \rightarrow \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} u_i \right) \geq \liminf_{n \rightarrow \infty} \frac{1}{2n} \cdot \sum_{i=0}^{n-1} (v_i - u_i) \geq 0.$$

Hence the strategy that corresponds to τ_1^* in the simulation game is a witness that there is no wise winning strategy for the challenger, i.e., B simulates A .

2. We now show that if the maximal value that the max-player can ensure is negative, then B does not simulate A . In this case, we fix a pure memoryless optimal strategy

τ_2^* for the min-player, and let us refer to the graph after fixing τ_2^* as $\mathcal{G}(A, B)_{\tau_2^*}$. Since the maximal value that the max-player can ensure is negative and τ_2^* is an optimal strategy, it follows that the sum of the weights of all cycles C reachable from q_I in $\mathcal{G}(A, B)_{\tau_2^*}$ is negative (i.e., the sum of weights by γ_1 exceeds the sum weights by γ_2 in C). By arguments similar to Theorem 5, it follows that given the strategy τ_2^* , for all strategies τ_1 of the max-player, the value of the resulting play in A exceeds the value of the resulting play in B . That is, the strategy that corresponds to τ_2^* in the simulation game is a (not necessarily blind) winning strategy for the challenger. It follows that B does not simulate A .

It follows from above that the maximal value that the max-player can enforce in $\mathcal{G}(A, B)$ is nonnegative if and only if B simulates A . The result then follows from the fact that the maximal value of perfect-information limit-average games can be decided in $\text{NP} \cap \text{coNP}$.

Finally, we consider the case of Disc-automata. Let $A = \langle Q_1, q_I^1, \Sigma, \delta_1, \gamma_1 \rangle$ and $B = \langle Q_2, q_I^2, \Sigma, \delta_2, \gamma_2 \rangle$ be two Disc-automata (with rational discount factor λ). We construct the discounted game $\mathcal{G}(A, B) = \langle Q_{\min}, Q_{\max}, q_I, E, \gamma \rangle$ with discount factor $\lambda' = \sqrt{\lambda}$ where:

- $Q_{\min} = Q_1 \times Q_2$;
- $Q_{\max} = Q_1 \times Q_2 \times \Sigma$;
- $q_I = (q_I^1, q_I^2)$;
- $E = \{((q_1, q_2), (q_1', q_2, \sigma)) \mid (q_1, \sigma, q_1') \in \delta_1\} \cup \{((q_1, q_2, \sigma), (q_1, q_2')) \mid (q_2, \sigma, q_2') \in \delta_2\}$.
- γ assigns $-\gamma_1(q_1, \sigma, q_1')$ to each $((q_1, q_2), (q_1', q_2, \sigma)) \in E$, and $\frac{1}{\lambda'} \cdot \gamma_2(q_2, \sigma, q_2')$ to each $((q_1, q_2, \sigma), (q_1, q_2')) \in E$.

It is easy to see that the maximal value that the max-player can enforce in this game is nonnegative if and only if B simulates A . Essentially, this is because if the sequence of weights in the play of the game $\mathcal{G}(A, B)$ is $-u_0, \frac{v_0}{\lambda'}, -u_1, \frac{v_1}{\lambda'}, \dots$, then its λ' -discounted sum is $(v_0 - u_0) + \lambda'^2 \cdot (v_1 - u_1) + \lambda'^4 \cdot (v_2 - u_2) + \dots$, that is $(v_0 - u_0) + \lambda \cdot (v_1 - u_1) + \lambda^2 \cdot (v_2 - u_2) + \dots$ which is nonnegative iff $\text{Disc}_\lambda(v_i) \geq \text{Disc}_\lambda(u_i)$.

Now, we show that deciding whether the value of the game $\mathcal{G}(A, B)$ is nonnegative can be done in $\text{NP} \cap \text{coNP}$. This would be straightforward if λ' was rational, but $\lambda' = \sqrt{\lambda}$ can be irrational even if λ is rational. It is known that perfect-information discounted games admit pure memoryless optimal strategies, and the pure memoryless optimal strategies serve as polynomial witnesses for the $\text{NP} \cap \text{coNP}$ procedure. To complete the $\text{NP} \cap \text{coNP}$ result we need to present polynomial-time verification procedure for graphs (i.e., game graphs after a pure memoryless strategy for a player is fixed). Suppose a pure memoryless strategy for one of the player (say the min player) is fixed, and then we do the following polynomial-time check: we construct a weighted graph from $\mathcal{G}(A, B)$ and the fixed strategy for player min by first removing the edges that are not played by the strategy, and then removing the min-states and replacing every path of length 2 between max-states by a direct edge, weighted $v - u$ if the corresponding two edges in the game were labeled $-u$ and $\frac{v}{\lambda'}$, and then solving the resulting graph as a λ -discounted graph (in polynomial-time [And06]). The dual construction can be done

in a similar fashion when a strategy for player max is fixed. Since checking if the minimal (or maximal) value of a discounted graph (with rational discount) is nonnegative can be done in polynomial time, we get an NP (or coNP) procedure. ■

5 The Expressive Power of Weighted Automata

We study the expressiveness of different classes weighted automata over infinite words by comparing the quantitative languages they can define. For this purpose, we show the existence and non-existence of translations between classes of finite and weighted automata. We will use the following definition. A class \mathcal{C} of finite automata *can be weakly reduced* to a class \mathcal{C}' of weighted automata if for every $A \in \mathcal{C}$ there exists an $A' \in \mathcal{C}'$ such that $\inf_{w \in L_A} L_{A'}(w) > \sup_{w \notin L_A} L_{A'}(w)$.

5.1 Positive Reducibility Results

We start with the positive results about the existence of reductions between various classes of weighted automata, most of which can be obtained by generalizing corresponding results for finite automata. Our results also hold if we allow transition weights to be irrational numbers.

First, it is clear that Büchi and coBüchi automata can be reduced to LimSup- and LimInf-automata, respectively. In addition, we have the following results.

Theorem 9. (i) Sup-automata can be determinized in $O(2^n)$ time; (ii) LimInf-automata can be determinized in $O(m \cdot 2^n)$ time. (iii) Deterministic Sup-automata can be reduced to deterministic LimInf-, to deterministic LimSup-, and to deterministic LimAvg-automata, all in $O(n \cdot m)$ time. (iv) LimInf-automata can be reduced to LimSup- and to LimAvg-automata, both in $O(n \cdot m)$ time.

Proof. (Sketch). (i) Given a Sup-automaton $A = \langle Q, q_I, \Sigma, \delta, \gamma \rangle$, we construct a deterministic Sup-automaton $A_D = \langle Q_D, q_I^D, \Sigma, \delta_D, \gamma_D \rangle$ such that $L_{A_D} = L_A$, using a subset construction:

- $Q_D = 2^Q$;
- $q_I^D = \{q_I\}$;
- δ_D contains all the transitions (s, σ, s') such that $\sigma \in \Sigma$ and $s' = \{q' \in Q \mid \exists q \in s : (q, \sigma, q') \in \delta\}$;
- γ_D assigns to $(s, \sigma, s') \in \delta_D$ the weight $v = \max\{\gamma(q, \sigma, q') \mid q \in s, q' \in s' \text{ and } (q, \sigma, q') \in \delta\}$.

(ii) Given a LimInf-automaton $A = \langle Q, q_I, \Sigma, \delta, \gamma \rangle$, we construct a deterministic LimInf-automaton $A_D = \langle Q_D, q_I^D, \Sigma, \delta_D, \gamma_D \rangle$ such that $L_{A_D} = L_A$. Let $v_1 < v_2 < \dots < v_k$ be the weights that appear on transitions of A . Define:

- $Q_D = \{(t_1, \dots, t_k) \mid t_i \subseteq Q \text{ for } i = 1, \dots, k\}$. Intuitively, A_D keeps k copies of the classical subset construction for finite automata, one for each weight in A . However, the transitions from a set t_i are limited to those with a weight at least v_i . Therefore, t_1 corresponds exactly to the subset construction, and never gets empty (since A is total). If t_i gets empty (for $i \geq 2$), it means that all the runs over the finite prefix of the input word that we have read contain a weight less than v_i ;
- $q_I^D = (q_I, \dots, q_I)$;
- δ_D contains all the transitions $((t_1, \dots, t_k), \sigma, (t'_1, \dots, t'_k))$ such that $\sigma \in \Sigma$ and for all $1 \leq i \leq k$,
 - if $t_i \neq \emptyset$, then $t'_i = \{q' \in Q \mid \exists q \in t_i : (q, \sigma, q') \in \delta \wedge \gamma(q, \sigma, q') \geq v_i\}$;
 - if $t_i = \emptyset$, then $t'_i = \{q' \in Q \mid \exists q \in t_1 : (q, \sigma, q') \in \delta \wedge \gamma(q, \sigma, q') \geq v_i\}$;
 When a set t_i is empty, it is initiated with the set of states from the classical subset construction, namely t_1 ;
- γ_D assigns to $((t_1, \dots, t_k), \sigma, (t'_1, \dots, t'_k)) \in \delta_D$ the weight v_m where $m = \max\{j \mid t'_j \neq \emptyset\}$. Intuitively, the value of the input word is at least v_i if and only if the set t_i never gets empty from some point on.

(iii) Given a deterministic Sup-automaton $A = \langle Q, q_I, \Sigma, \delta, \gamma \rangle$, we construct a deterministic LimInf-automaton $A_D = \langle Q_D, q_I^D, \Sigma, \delta_D, \gamma_D \rangle$ such that $L_{A_D} = L_A$. Let $v_1 < v_2 < \dots < v_k$ be the weights that appear on transitions of A . Define:

- $Q_D = Q \times \{1, \dots, k\}$;
- $q_I^D = (q_I, 1)$;
- δ_D contains all the transitions $((q, i), \sigma, (q', i'))$ such that $(q, \sigma, q') \in \delta$ and $i' = \max\{i, k\}$ where k is such that $v_k = \gamma(q, \sigma, q')$;
- $\gamma_D((q, i), \sigma, (q', i')) = v_i$ for all $((q, i), \sigma, (q', i')) \in \delta_D$.

To show that deterministic Sup-automata are reducible to deterministic LimSup-automata (resp. to deterministic LimAvg-automata), we use the same automaton A_D interpreted as a LimSup- (resp. LimAvg-) automaton.

(iv) The reduction from LimInf- to LimSup-automata (and to LimAvg-automata) essentially consists of guessing a position i and a transition weight v such that only weights greater than v are seen after position i . Once the guess is made, all transitions have weight v . Given a LimInf-automaton A we present an equivalent LimAvg-automaton B . Let $v_1 < v_2 < v_3 < \dots < v_k$ be the set of weights of A . The automaton B is obtained as follows: we make k copies A_1, A_2, \dots, A_k of the automaton A ; in automaton A_i we only allow transitions of A of weights at least v_i and assign each of them weight v_i . We start in automaton A_1 and at any point can choose to stay in A_i or choose to move to any of the copies A_{i+1}, \dots, A_k . Transitions that are not allowed in B leads to sink with weight v_1 . ■

All reducibility relationships are summarized in Fig. 7 and Table 3, where the notation nyW is used to denote the classes of automata that are determinizable.

5.2 Negative Reducibility Results

We show that all other reducibility relationships do not hold. The most important results in this section show that (i) deterministic coBüchi automata cannot be reduced

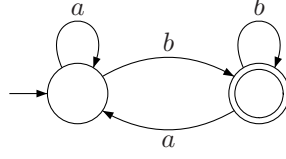


Fig. 3. A deterministic coBüchi automaton.

to deterministic LimAvg-automata, deterministic Büchi automata cannot be reduced to LimAvg-automata, and (ii) neither LimAvg- nor Disc-automata can be determinized. Over the alphabet $\hat{\Sigma} = \{a, b\}$, we use in the sequel the boolean languages L_F , which contains all infinite words with finitely many a 's, and L_I , which contains all infinite words with infinitely many a 's.

The classical proof that deterministic coBüchi automata cannot be reduced to deterministic Büchi automata can be adapted to show the following theorem.

Theorem 10. *Deterministic coBüchi automata cannot be reduced to deterministic LimSup-automata.*

Since deterministic LimAvg- and deterministic Disc-automata can define quantitative languages whose range is infinite, while LimSup-automata cannot, we obtain the following result (the proof is given in the appendix).

Theorem 11. *Deterministic LimAvg-automata and deterministic Disc-automata cannot be reduced to LimSup-automata.*

Remark. For the automaton A of the Theorem 11, if we consider the language $L = \{w \in \Sigma^\omega \mid A(w) = 1\}$, then there is no NBW or NCW that accepts the language L ; this is because it is known from [Cha07b] that the set L is not ω -regular and is complete for the third level of Borel hierarchy.

The next theorem shows that nondeterministic LimAvg-automata are strictly more expressive than their deterministic counterpart. Theorem 13 will show that the expressive powers of LimAvg- and LimSup-automata are incomparable.

Theorem 12. *Deterministic coBüchi automata cannot be weakly reduced to deterministic LimAvg-automata, and therefore they cannot be reduced to deterministic LimAvg-automata. LimAvg-automata cannot be determinized.*

Proof. Consider the language L_F of finitely many a 's, which is the language defined by the DCW shown in Fig. 3. It is also easy to see that the NLAW shown in Fig. 4 defines L_F . We show that L_F cannot be defined by any DLAW to prove the desired claims. By contradiction, assume that A is a DLAW with set of states Q and the initial state q_I that defines L_F . We assume without loss of generality that every state $q \in Q$ is reachable from q_I by a finite word w_q .

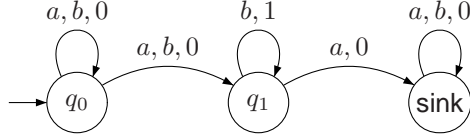


Fig. 4. A nondeterministic limit-average automaton.

Let $\alpha = \inf_{w \in L_F} L_A(w)$. We claim that all b -cycles (a b -cycle is a cycle in A that can be executed with only b 's) must be such that the average of the weights on the cycle is at least α . Indeed, if there is a b -cycle C in A with average weights less than α , then consider a state $q \in C$ and the word $w = w_q \cdot b^\omega$. We have $L_A(w) < \alpha$. Since $w = w_q \cdot b^\omega \in L_F$, this contradicts that $\alpha = \inf_{w \in L_F} L_A(w)$.

We now show that for all $\epsilon > 0$, there exists $w' \notin L_F$ such that $L_A(w') \geq \alpha - \epsilon$. Fix $\epsilon > 0$. Let $\beta = \max_{q, q' \in Q, \sigma \in \{a, b\}} |\gamma(q, \sigma, q')|$. Let $j = \lceil \frac{6 \cdot |Q| \cdot \beta}{\epsilon} \rceil$, and consider the word $w_\epsilon = (b^j \cdot a)^\omega$. A lower bound on the average of the weights in the unique run of A over $(b^j \cdot a)^\omega$ is as follows: it can have a prefix of length at most $|Q|$ whose sum of weights is at least $-|Q| \cdot \beta$, then it goes through b -cycles for at least $j - 2 \cdot |Q|$ steps with sum of weights at least $(j - 2 \cdot |Q|) \cdot \alpha$ (since all b -cycles have average weights at least α), then again a prefix of length at most $|Q|$ without completing the cycle (with sum of weights at least $-|Q| \cdot \beta$), and then weight for a is at least $-\beta$. Hence the average is at least

$$\frac{(j - 2 \cdot |Q|) \cdot \alpha - 2 \cdot |Q| \cdot \beta - \beta}{j + 1} \geq \alpha - \frac{6 \cdot |Q| \cdot \beta}{j} \geq \alpha - \epsilon;$$

we used above that $|\alpha| \leq \beta$, and by choice of j we have $\frac{6 \cdot |Q| \cdot \beta}{j} \leq \epsilon$. Hence we have $L_A(w_\epsilon) \geq \alpha - \epsilon$. Since $\epsilon > 0$ is arbitrary, and $w_\epsilon \notin L_F$, we have $\sup_{w \notin L_F} L_A(w) \geq \alpha = \inf_{w \in L_F} L_A(w)$. This establishes a contradiction, and thus A cannot exist. The desired result follows. \blacksquare

Theorem 13. *Deterministic Büchi automata cannot be weakly reduced to LimAvg-automata, and therefore they cannot be reduced to LimAvg-automata.*

Proof. We consider the language L_I of infinitely many a 's which is accepted by the DBW shown in Fig. 5.

By contradiction, assume that A is a NLAW with set of states Q and initial state q_I that defines L_I . We assume without loss of generality that every state $q \in Q$ is reachable from q_I by a finite word w_q .

Let $\alpha = \sup_{w \notin L_I} L_A(w)$, and $\beta = \max_{q, q' \in Q, \sigma \in \{a, b\}} |\gamma(q, \sigma, q')|$. We claim that all b -cycles C in A must have average weights at most α ; otherwise, consider a state $q \in C$ and the word $w = w_q \cdot b^\omega$, we have $L_A(w) > \alpha$ which contradicts that $\alpha = \sup_{w \notin L_I} L_A(w)$.

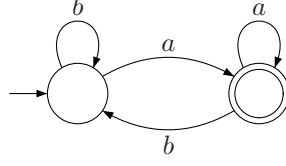


Fig. 5. A deterministic Büchi automaton.

We now show that for all $\epsilon > 0$, there exists $w \in L_I$ such that $L_A(w') \leq \alpha + \epsilon$. Fix $\epsilon > 0$. Let $j = \lceil \frac{3 \cdot |Q| \cdot \beta}{\epsilon} \rceil$, and consider the word $w_\epsilon = (b^j \cdot a)^\omega$. An upper bound on the average of the weights in any run of A over $(b^j \cdot a)$ is as follows: it can have a prefix of length at most $|Q|$ with the sum of weights at most $|Q| \cdot \beta$, then it follows (possibly nested) b -cycles³ for at most j steps with sum of weights at most $j \cdot \alpha$ (since all b -cycles have average weights at most α), then again a prefix of length at most $|Q|$ without completing a cycle (with sum of weights at most $|Q| \cdot \beta$), and then weight for a is at most β . So, for any run of A over $w_\epsilon = (b^j \cdot a)^\omega$, the average weight is at most

$$\frac{j \cdot \alpha + 2 \cdot |Q| \cdot \beta + \beta}{j + 1} \leq \alpha + \frac{3 \cdot |Q| \cdot \beta}{j} \leq \alpha + \epsilon$$

Hence we have $L_A(w_\epsilon) \leq \alpha + \epsilon$. Since $\epsilon > 0$ is arbitrary, and $w_\epsilon \in L_I$, we have $\inf_{w \in L_I} L_A(w) \leq \alpha = \sup_{w \notin L_I} L_A(w)$. The desired result follows. ■

None of the weighted automata we consider can be reduced to Disc-automata (Theorem 14), and Disc-automata cannot be reduced to any of the other classes of weighted automata (Theorem 15, and also Theorem 11).

Theorem 14. *Deterministic coBüchi automata and deterministic Büchi automata cannot be weakly reduced to Disc-automata, and therefore they cannot be reduced to Disc-automata. Also deterministic Sup-automata cannot be reduced to Disc-automata.*

The proofs of Theorem 14 and 15 are based on the property that the value assigned by a Disc-automaton to an infinite word depends essentially on a finite prefix, in the sense that the values of two words become arbitrarily close when they have sufficiently long common prefixes. In other words, the quantitative language defined by a discounted-sum automaton is a continuous function in the Cantor topology. In contrast, for the other classes of weighted automata, the value of an infinite word depends essentially on its tail.

Proof of Theorem 14. First, we show that deterministic coBüchi automata cannot be weakly reduced to Disc-automata. Consider the language L_f of finitely many a 's. The language L_f is accepted by the DCW shown in Fig. 3.

³ Since A is nondeterministic, a run over b^j may have nested cycles. We can decompose the run by repeatedly eliminating the innermost cycles.

We show that L_f is not weakly reducible to any NDIW. By contradiction, assume that there exists a NDIWA such that $\alpha = \inf_{w \in L_f} L_A(w) > \sup_{w \notin L_f} L_A(w) = \beta$. Then, $L_A(a^i b^\omega) \geq \alpha$ for all $i \geq 0$. So, for all $\epsilon > 0$, there exists $i \geq 0$ such that A has a run over a^i with value at least $\alpha - \epsilon$. Therefore $L_A(a^\omega) \geq \alpha - 2\epsilon$. Since this holds for all $\epsilon > 0$, we have $L_A(a^\omega) \geq \alpha$. Similarly, $L_A(b^i a^\omega) \leq \beta$ for all $i \geq 0$, and for all $\epsilon > 0$, there exists $i \geq 0$ such that A has all its runs over b^i with value at most $\beta + \epsilon$. Therefore $L_A(b^\omega) \leq \beta + 2\epsilon$. Since this holds for all $\epsilon > 0$, we have $L_A(b^\omega) \leq \beta$. Since $a^\omega \notin L_f$ and $b^\omega \in L_f$, this contradicts that $\alpha > \beta$.

Second, we show that deterministic Büchi automata cannot be weakly reduced to Disc-automata. We consider the language L_I of infinitely many a 's. The DBW shown in Fig. 5 accepts L_I . We now show that L_I is not weakly reducible to NDIW.

By contradiction, assume that there exists a NDIWA such that $\alpha = \inf_{w \in L_I} L_A(w) > \sup_{w \notin L_I} L_A(w) = \beta$. Then, $L_A(b^i a^\omega) \geq \alpha$ for all $i \geq 0$. So, for all $\epsilon > 0$, there exists $i \geq 0$ such that A has a run over b^i with value at least $\alpha - \epsilon$. Therefore $L_A(b^\omega) \geq \alpha - 2\epsilon$. Since this holds for all $\epsilon > 0$, we have $L_A(b^\omega) \geq \alpha$. Similarly, $L_A(a^i b^\omega) \leq \beta$ for all $i \geq 0$, and for all $\epsilon > 0$, there exists $i \geq 0$ such that A has all its runs over a^i with value at most $\beta + \epsilon$. Therefore $L_A(a^\omega) \leq \beta + 2\epsilon$. Since this holds for all $\epsilon > 0$, we have $L_A(a^\omega) \leq \beta$. Since $a^\omega \in L_I$ and $b^\omega \notin L_I$, this contradicts that $\alpha > \beta$.

Third, we show that Sup-automata cannot be reduced to Disc-automata. Consider the DSUPW A (shown in Fig. 8) that consists of a single self-loop state with weight 1 for a and 0 for b .

Assume that there exists a NDIW B such that for all $w \in \Sigma^\omega$ we have $L_A(w) = L_B(w)$. For each $i \geq 0$, consider the word $w_i = b^i a^\omega$. We have $L_B(w_i) = L_A(w_i) = 1$, and thus for all $\epsilon > 0$, there exists $i \geq 0$ such that B has a run over b^i with value at least $1 - \epsilon$. Therefore $L_B(b^\omega) \geq 1 - 2\epsilon$. Since this holds for all $\epsilon > 0$, we have $L_B(b^\omega) \geq 1$. However, $L_A(b^\omega) = 0$ which contradicts that $L_A(w) = L_B(w)$ for all $w \in \Sigma^\omega$. ■

Theorem 15. *Deterministic Disc-automata cannot be reduced to LimAvg-automata.*

Proof. Consider the DDIW A (shown in Fig. 8) that consists of a single self-loop state with weight 1 for a and 0 for b .

Assume that there exists a NLAW B such that for all $w \in \Sigma^\omega$ we have $L_A(w) = L_B(w)$. For each $i \geq 0$, consider the finite word $w_i = a^i$ and let s_i be the set of states of B in which can be the last state of a run of B over w_i . Since B is finite, there exist $j \neq k$ such that $s_j = s_k$. Therefore, $L_B(w_j b^\omega) = L_B(w_k b^\omega)$. However, $L_A(w_j b^\omega) = \frac{1-\lambda^j}{1-\lambda}$ and $L_A(w_k b^\omega) = \frac{1-\lambda^k}{1-\lambda}$ and thus $L_A(w_j b^\omega) \neq L_A(w_k b^\omega)$ which establishes a contradiction. ■

The next result shows that discounted-sum automata cannot be determinized. Consider the nondeterministic discounted-sum automaton N over the alphabet $\hat{\Sigma} = \{a, b\}$ shown in Fig. 6. The automaton N computes the maximum of the discounted sum of

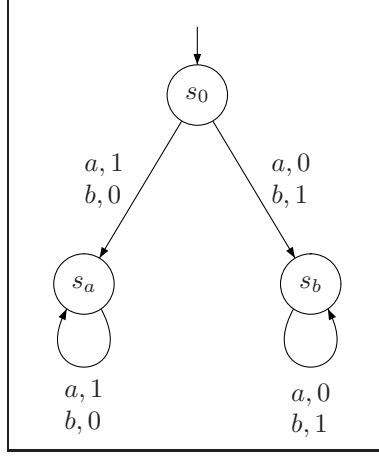


Fig. 6. The automaton N .

a 's and b 's. Formally, given a (finite or infinite) word $w = w_0 w_1 \dots \in \hat{\Sigma}^* \cup \hat{\Sigma}^\omega$, let

$$v_a(w) = \sum_{i|w_i=a}^{|w|} \lambda^i \quad \text{and} \quad v_b(w) = \sum_{i|w_i=b}^{|w|} \lambda^i$$

be the λ -discounted sum of all a 's (resp. b 's) in w . Then $L_N(w) = \max\{v_a(w), v_b(w)\}$ for all infinite words $w \in \hat{\Sigma}^\omega$. We show that N cannot be determinized for some discount factors λ . The proof uses a sequence of intermediate lemmas.

For $\sigma \in \hat{\Sigma}$, let $\bar{\sigma} = a$ if $\sigma = b$, and $\bar{\sigma} = b$ if $\sigma = a$. We say that an infinite word $w \in \hat{\Sigma}^\omega$ *prefers* $\sigma \in \hat{\Sigma}$ if $v_\sigma(w) > v_{\bar{\sigma}}(w)$.

Lemma 4. *For all $0 < \lambda < 1$, all $w \in \hat{\Sigma}^*$, and all $\sigma \in \hat{\Sigma}$, there exists $w' \in \hat{\Sigma}^\omega$ such that $w \cdot w'$ prefers σ iff $v_\sigma(w \cdot \sigma^\omega) > v_{\bar{\sigma}}(w \cdot \sigma^\omega)$.*

Proof. Assume that $w \cdot w'$ strictly prefers σ . Then $v_\sigma(w \cdot \sigma^\omega) \geq v_\sigma(w \cdot w') > v_{\bar{\sigma}}(w \cdot w') \geq v_{\bar{\sigma}}(w \cdot \sigma^\omega)$. The reverse direction is trivial. ■

We say that a finite word $w \in \hat{\Sigma}^*$ is *ambiguous* if there exist two infinite words $w'_a, w'_b \in \hat{\Sigma}^\omega$ such that $w \cdot w'_a$ prefers a and $w \cdot w'_b$ prefers b .

Lemma 5. *For all $0 < \lambda < 1$ and $w \in \hat{\Sigma}^*$, the word w is ambiguous iff $|v_a(w) - v_b(w)| < \frac{\lambda^{|w|}}{1-\lambda}$.*

Proof. By Lemma 4, w is ambiguous if and only if $v_a(w \cdot a^\omega) > v_b(w \cdot a^\omega)$ and $v_b(w \cdot b^\omega) > v_a(w \cdot b^\omega)$, that is

$$v_a(w) + \frac{\lambda^{|w|}}{1-\lambda} > v_b(w) \quad \text{and} \quad v_b(w) + \frac{\lambda^{|w|}}{1-\lambda} > v_a(w)$$

Intuitively, ambiguous words are problematic for a deterministic automaton because it cannot decide which one of the two functions v_a and v_b to choose. ■

Lemma 6. *For all $\frac{1}{2} < \lambda < 1$, there exists an infinite word $\hat{w} \in \hat{\Sigma}^\omega$ such that every finite prefix of \hat{w} is ambiguous.*

Proof. We construct $\hat{w} = w_1 w_2 \dots$ inductively as follows. First, let $w_1 = a$ which is an ambiguous word for all $\lambda > \frac{1}{2}$ (Lemma 5). Assume that $w_1 \dots w_i$ is ambiguous for all $1 \leq i \leq k$, that is $|x_i| < \frac{\lambda^i}{1-\lambda}$ where $x_i = v_a(w_1 \dots w_i) - v_b(w_1 \dots w_i)$ (Lemma 5). We take $w_{k+1} = a$ if $x_k < 0$, and $w_{k+1} = b$ otherwise. Let us show that $|x_{k+1}| < \frac{\lambda^{k+1}}{1-\lambda}$. We have $|x_{k+1}| = ||x_k| - \lambda^k|$, and thus we need to show that $|x_k| - \lambda^k < \frac{\lambda^{k+1}}{1-\lambda}$ and $-|x_k| + \lambda^k < \frac{\lambda^{k+1}}{1-\lambda}$ knowing that $|x_k| < \frac{\lambda^k}{1-\lambda}$. It suffices to show that

$$\frac{\lambda^k}{1-\lambda} \leq \lambda^k + \frac{\lambda^{k+1}}{1-\lambda} \quad \text{and} \quad \lambda^k - \frac{\lambda^{k+1}}{1-\lambda} < 0.$$

In other words, it suffices that $1 \leq 1 - \lambda + \lambda$ and $1 - \lambda - \lambda < 0$, which is true for all $\lambda > \frac{1}{2}$. ■

The word \hat{w} constructed in Lemma 6 could be harmless for a deterministic automaton if some kind of periodicity is encountered in \hat{w} . We make this notion formal by defining $\text{diff}(w) = \frac{v_a(w) - v_b(w)}{\lambda^{|w|}}$ for all finite words $w \in \hat{\Sigma}^*$. It can be shown that if the set $R_\lambda = \{\text{diff}(w) \mid w \in \hat{\Sigma}^*\} \cap (\frac{-1}{1-\lambda}, \frac{1}{1-\lambda})$ is finite, then the automaton N can be determinized (see Appendix B), where (a, b) denotes the open interval between two reals a and b with $a < b$. Lemma 7 shows that this is also a necessary condition.

Lemma 7. *For all $0 < \lambda < 1$, if the set R_λ is infinite, then there exists no deterministic Disc-automaton D such that $L_D = L_N$.*

Proof. By contradiction, assume that R_λ is infinite and there exists a DDIW D such that $L_D = L_N$. For all $w \in \hat{\Sigma}^*$, let $\text{Post}(w)$ be the (unique) state reached in D after reading w . We show that for all words $w_1, w_2 \in \hat{\Sigma}^*$ such that $\text{diff}(w_1), \text{diff}(w_2) \in R_\lambda$, if $\text{diff}(w_1) \neq \text{diff}(w_2)$, then $\text{Post}(w_1) \neq \text{Post}(w_2)$. Therefore D cannot have finitely many states.

We show this by contradiction. Assume that $\text{Post}(w_1) = \text{Post}(w_2)$. Then w_1 and w_2 are ambiguous by Lemma 5 since $\text{diff}(w_1), \text{diff}(w_2) \in R_\lambda$. By Lemma 4, we thus have for $i = 1, 2$,

$$L_N(w_i \cdot a^\omega) = v_a(w_i) + \frac{\lambda^{|w_i|}}{1-\lambda} \quad \text{and} \quad L_N(w_i \cdot b^\omega) = v_b(w_i) + \frac{\lambda^{|w_i|}}{1-\lambda}.$$

On the other hand, since $\text{Post}(w_1) = \text{Post}(w_2)$, there exist $v_1, v_2, K_a, K_b \in \mathbb{R}$ such that for $i = 1, 2$,

$$L_D(w_i \cdot a^\omega) = v_i + \lambda^{|w_i|} \cdot K_a \quad \text{and} \quad L_D(w_i \cdot b^\omega) = v_i + \lambda^{|w_i|} \cdot K_b.$$

Since $L_D = L_N$, this entails that $L_D(w_i \cdot a^\omega) - L_D(w_i \cdot b^\omega) = L_N(w_i \cdot a^\omega) - L_N(w_i \cdot b^\omega)$, and therefore

$$\frac{v_a(w_1) - v_b(w_1)}{\lambda^{|w_1|}} = K_a - K_b = \frac{v_a(w_2) - v_b(w_2)}{\lambda^{|w_2|}}$$

which yields a contradiction (to the fact that $\text{diff}(w_1) \neq \text{diff}(w_2)$). \blacksquare

We are now ready to prove the following theorem.

Theorem 16. *Disc-automata cannot be determinized.*

Proof. Let λ^* be a non-algebraic number in the open interval $(\frac{1}{2}, 1)$. Then, we show that the set R_{λ^*} is infinite, which establishes the theorem by Lemma 7.

By Lemma 5 and Lemma 6, there exist infinitely many finite words $w \in \hat{\Sigma}^*$ such that $\text{diff}(w) \in R_{\lambda^*}$. Since λ^* is not algebraic, the polynomial equation $\text{diff}(w_1) = \text{diff}(w_2)$ cannot hold for $w_1 \neq w_2$. Therefore, R_{λ^*} is infinite. \blacksquare

By a careful analysis of the shape of the family of polynomial equations in the above proof (see Lemma 8), we can show that the automaton N cannot be determinized for any rational value of λ greater than $\frac{1}{2}$.

Lemma 8. *For all finite words $w_1, w_2 \in \Sigma^*$ with $w_1 \neq w_2$, the polynomial equation $\text{diff}(w_1) = \text{diff}(w_2)$ in variable λ has no rational solution in $[\frac{1}{2}, 1]$.*

Proof. First, consider a polynomial $f(x) = c_0 + c_1 \cdot x + \dots + c_n \cdot x^n$ with integer coefficients, $c_0 \neq 0$ and $c_n \neq 0$. If $f(\frac{p}{q}) = 0$ for some mutually prime integers p and q , then we have

$$c_0 \cdot q^n + c_1 \cdot p \cdot q^{n-1} + \dots + c_{n-1} \cdot p^{n-1} \cdot q + c_n \cdot p^n = 0.$$

The first term in this sum must be a multiple of p since the rest of the sum is divisible by p , and analogously the last term must be a multiple of q . Since $\frac{p}{q}$ is irreducible, it must be that p divides c_0 and q divides c_n .

Now, consider the equation $\text{diff}(w_1) = \text{diff}(w_2)$:

$$\frac{r_a(w_1) - r_b(w_1)}{\lambda^{|w_1|}} = \frac{r_a(w_2) - r_b(w_2)}{\lambda^{|w_2|}}.$$

Assume w.l.o.g that $n = |w_2| - |w_1| \geq 0$. We get the equation

$$\lambda^n (r_a(w_1) - r_b(w_1)) = r_a(w_2) - r_b(w_2).$$

If $n > 0$, then the coefficient of the term of degree zero in the above equation is either 1 or -1 , and by the above argument, the only rational solutions it can have are 1 or -1 . If $n = 0$, then it is easy to see that at least one coefficients is not zero (since $w_1 \neq w_2$) and every non-zero coefficient is either 2 or -2 . Therefore, as above the only possible rational solutions are 1 or -1 . The result follows. \blacksquare

Reducibility		boolean			quantitative							
		P^{CW}	DBW	NBW	R^{SUPW}	L^{LiW}	DLsW	NLsW	DLAW	NLAW	DDiW	NDiW
boolean	P^{CW}	·	×	✓	×	✓	×	✓	×	✓	×	×
	DBW	×	·	✓	×	×	×	✓	×	×	×	×
	NBW	×	×	·	×	×	×	✓	×	×	×	×
quantitative	P^{SUPW}	×	·	✓	✓	✓	✓	✓	×	×		
	L^{LiW}		×	·	×	✓	×	✓	×	×		
	DLsW		×	×	·	✓	×	×	×	×		
	NLsW		×	×	×	·	×	×	×	×		
	DLAW		×	×	×	×	·	✓	×	×		
	NLAW		×	×	×	×	×	·	×	×		
	DDiW		×	×	×	×	×	×	·	✓		
	NDiW		×	×	×	×	×	×	×	·		

Table 3. Reducibility relation. \mathcal{C} is reducible to \mathcal{C}' if the entry $R(\mathcal{C}, \mathcal{C}')$ is $\sqrt{\cdot}$.

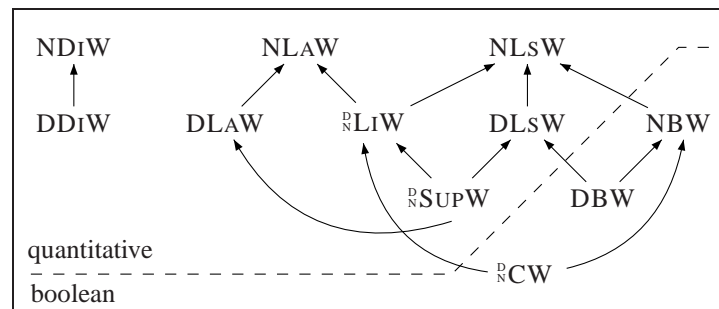


Fig. 7. Reducibility relations: a class \mathcal{C} of automata can be reduced to \mathcal{C}' iff $\mathcal{C} \rightarrow^* \mathcal{C}'$.

References

- [And06] D. Andersson. An improved algorithm for discounted payoff games. In *ESSLI Student Session*, pages 91–98, 2006.
- [CCH⁺05] A. Chakrabarti, K. Chatterjee, T. A. Henzinger, O. Kupferman, and R. Majumdar. Verifying quantitative properties using bound functions. In *CHARME*, LNCS 3725, pages 50–64. Springer, 2005.
- [CdHS03] A. Chakrabarti, L. de Alfaro, T. A. Henzinger, and M. Stoelinga. Resource interfaces. In *EMSOFT*, LNCS 2855, pages 117–133. Springer, 2003.
- [CGH⁺08] K. Chatterjee, A. Ghosal, T. A. Henzinger, D. Iercan, C. Kirsch, C. Pinello, and A. Sangiovanni-Vincentelli. Logical reliability of interacting real-time tasks. In *DATE*. ACM, 2008.

- [Cha07a] K. Chatterjee. Concurrent games with tail objectives. *Theoretical Computer Science*, 388:181–198, 2007.
- [Cha07b] K. Chatterjee. *Stochastic ω -Regular Games*. PhD thesis, University of California, Berkeley, 2007.
- [Cha08] Krishnendu Chatterjee. Linear time algorithm for weak parity games. *CoRR*, abs/0805.1391, 2008.
- [Con92] Anne Condon. The complexity of stochastic games. *Information and Computation*, 96:203–224, 1992.
- [dAHM03] L. de Alfaro, T. A. Henzinger, and R. Majumdar. Discounting the future in systems theory. In *ICALP*, LNCS 2719, pages 1022–1037. Springer, 2003.
- [DG07] M. Droste and P. Gastin. Weighted automata and weighted logics. *Theoretical Computer Science*, 380:69–86, 2007.
- [DK03] M. Droste and D. Kuske. Skew and infinitary formal power series. In *ICALP*, LNCS 2719, pages 426–438. Springer, 2003.
- [DKR08] M. Droste, W. Kuich, and G. Rahonis. Multi-valued MSO logics over words and trees. *Fundamenta Informaticae*, 84:305–327, 2008.
- [EJ91] E. A. Emerson and C. Jutla. Tree automata, mu-calculus and determinacy. In *FOCS*, pages 368–377. IEEE, 1991.
- [EM79] A. Ehrenfeucht and J. Mycielski. Positional strategies for mean payoff games. *International Journal of Game Theory*, 8:109–113, 1979.
- [FV97] J. Filar and K. Vrieze. *Competitive Markov Decision Processes*. Springer, 1997.
- [GC03] A. Gurfinkel and M. Chechik. Multi-valued model checking via classical model checking. In *CONCUR*, LNCS 2761, pages 263–277. Springer, 2003.
- [Gim06] H. Gimbert. *Jeux positionnels*. PhD thesis, Université Paris 7, 2006.
- [HKR97] T.A. Henzinger, O. Kupferman, and S.K. Rajamani. Fair simulation. In *CONCUR*, LNCS 1243, pages 273–287. Springer, 1997.
- [Kar78] R. M. Karp. A characterization of the minimum cycle mean in a digraph. *Discrete Mathematics*, 23(3):309–311, 1978.
- [KL07] O. Kupferman and Y. Lustig. Lattice automata. In *VMCAI*, LNCS 4349, pages 199–213. Springer, 2007.
- [KM05] D. Kirsten and I. Mäurer. On the determinization of weighted automata. *Journal of Automata, Languages and Combinatorics*, 10:287–312, 2005.
- [Kro92] D. Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. In *ICALP*, LNCS 623, pages 101–112. Springer, 1992.
- [KS86] W. Kuich and A. Salomaa. *Semirings, Automata, Languages*, volume 5 of *Monographs in Theoretical Computer Science. An EATCS Series*. Springer, 1986.
- [Moh97] M. Mohri. Finite-state transducers in language and speech processing. *Comp. Linguistics*, 23(2):269–311, 1997.
- [MS72] A. R. Meyer and L. J. Stockmeyer. The equivalence problem for regular expressions with squaring requires exponential space. In *FOCS*, pages 125–129. IEEE, 1972.
- [Paz71] A. Paz. *Introduction to probabilistic automata*. Computer Science and Applied Mathematics. Academic Press, New York, 1971.
- [Sch61] M. P. Schützenberger. On the definition of a family of automata. *Information and control*, 4:245–270, 1961.
- [Sha53] L. S. Shapley. Stochastic games. In *Proc. of the National Academy of Science USA*, volume 39, pages 1095–1100, 1953.
- [SVW87] A. P. Sistla, M. Y. Vardi, and P. Wolper. The complementation problem for Büchi automata with applications to temporal logic. *Theoretical Computer Science*, 49:217–237, 1987.
- [Tho97] W. Thomas. Languages, automata, and logic. In *Handbook of Formal Languages*, volume 3, Beyond Words, chapter 7, pages 389–455. Springer, 1997.

- [Wad84] W.W. Wadge. *Reducibility and Determinateness of Baire Spaces*. PhD thesis, UC Berkeley, 1984.

A Proofs

Proof of Theorem 1. We show that the quantitative language-inclusion problem is PSPACE-complete for Last- and Max-automata.

1. Given two Last-automata A and B , it is easy to construct (in polynomial time) for each weight v appearing in A or B the (boolean) finite automata $A^{\geq v}$ and $B^{\geq v}$ that accept the finite words with a value at least v according to L_A and L_B respectively. Then the quantitative language inclusion problem for A and B is equivalent to check that $L^f(A^{\geq v}) \subseteq L^f(B^{\geq v})$ for each weight v appearing in A or B , which can be done in polynomial space.

The hardness result is obtained by a straightforward reduction of the boolean language inclusion problem for finite automata which is PSPACE-COMplete.

2. Given two Sup-automata $A = \langle Q_1, q_I^1, \Sigma, \delta_1, \gamma_1 \rangle$, and $B = \langle Q_2, q_I^2, \Sigma, \delta_2, \gamma_2 \rangle$, we construct a (boolean) finite automaton C whose language is empty if and only if $L_A \subseteq L_B$. The (N)PSPACE algorithm will explore this automaton on-the-fly. We assume for $i = 1, 2$ that $\gamma_i(e) = \perp$ for all $e \notin \delta_i$, and let V_1, V_2 be the sets of weights appearing on transitions of A and B respectively. We define $C = \langle Q, q_I, \Sigma, \delta \rangle$ as follows:

- $Q = 2^{Q_1} \times \Gamma_1 \times 2^{Q_2} \times \Gamma_2$ where Γ_i is the set of functions $f : Q_i \rightarrow V_i$ for $i = 1, 2$;
- $q_I = (\{q_I^1\}, f_1, \{q_I^2\}, f_2)$ where $f_i(q) = \min(V_i)$ for all $q \in Q_i$ and $i = 1, 2$;
- δ contains all the tuples $\langle (s_1, f_1, s_2, f_2), \sigma, (s'_1, f'_1, s'_2, f'_2) \rangle$ such that (for $i = 1, 2$):
 - $s'_i = \{q' \in Q_i \mid \exists q \in s_i : (q, \sigma, q') \in \delta_i\}$;
 - for all $q' \in Q_i$, $f'_i(q') = \max\{f_i(q) + \gamma_i(q, \sigma, q') \mid q \in s_i\}$ with the assumptions that $\max \emptyset = \perp$ and $n + \perp = \perp$ for all $n \in \mathbb{Q}$;

The set of accepting states of C is $F_C = \{(s_1, f_1, s_2, f_2) \mid \exists q_1 \in s_1 \cdot \forall q_2 \in s_2 : f_1(q_1) > f_2(q_2)\}$. It is easy to see that $L^f(C) \neq \emptyset$ if and only if there is a finite word w such that $L_A(w) > L_B(w)$.

The hardness result is obtained by a straightforward reduction of the boolean language inclusion problem for finite automata which is PSPACE-COMplete. ■

Proof of Lemma 1. Let $V_1 = \{\gamma_1(e) \mid e \in \delta_1\}$ be the (finite) set of weights that occur on some transitions in A . The construction is as follows:

- $Q_2 = Q_1 \cup (Q_1 \times V_1)$;
- $q_I^2 = q_I^1$;
- $\delta_2 = \delta_1 \cup \{(q, \sigma, (q', v)) \mid (q, \sigma, q') \in \delta_1 \text{ and } \gamma_1(q, \sigma, q') = v\} \cup \{((q, v), \sigma, (q, v)) \mid q \in Q_1, v \in V_1 \text{ and } \sigma \in \Sigma\}$;
- $\gamma_2(e) = \gamma_1(e)$ for all $e \in \delta_1$, and $\gamma_2(q_2, \sigma, (q, v)) = v$ for all $q_2 \in Q_2$, $\sigma \in \Sigma$ and $(q, v) \in Q_1 \times V_1$.

For every run r_1 of A , we can easily construct a run r_2 of B on the same word such that $\text{Sup}(\gamma(r_1)) = \text{LimSup}(\gamma(r_2))$ by looping through a state of the form (q, v) where $v = \text{Sup}(\gamma(r_1))$ is the maximal value occurring in r_1 . Thus we have $L_A(w) \leq L_B(w)$ for all $w \in \Sigma^\omega$.

Similarly, for every run r_2 of B , we can construct a run r_1 of A on the same word such that $\text{LimSup}(\gamma(r_2)) \leq \text{Sup}(\gamma(r_1))$. Indeed, either r_2 is also a run of A or it is looping through a state $(q, v) \in Q_1 \times V_1$. In each case, r_2 has a finite prefix which can be executed in A and contains a transition with weight $v = \text{LimSup}(\gamma(r_2))$. We obtain r_1 by prolonging this prefix in A . Hence, we have $L_B(w) \leq L_A(w)$ for all $w \in \Sigma^\omega$. ■

Proof of Lemma 2. We construct $A^{\geq v} = \langle Q^{\geq v}, q_I^{\geq v}, \Sigma, \delta^{\geq v} \rangle$ as follows:

- $Q^{\geq v} = Q \times \{0, 1\}$;
- $q_I^{\geq v} = (q_I, 0)$;
- $\delta^{\geq v}$ contains all the tuples $((q, i), \sigma, (q', j))$ such that $(q, \sigma, q') \in \delta$ and
 - $i = 0$ and $j = 0$ and $\gamma(q, \sigma, q') < v$, or
 - $i = 0$ and $j = 1$ and $\gamma(q, \sigma, q') \geq v$, or
 - $i = 1$ and $j = 0$.
- $F^{\geq v} = Q \times \{1\}$.

It is easy to see that for all infinite words $w \in \Sigma^\omega$, A has a run r over w with $\text{Val}(\gamma(r)) \geq v$ if and only if $A^{\geq v}$ has an accepting run over w . ■

Theorem 17. *The quantitative universality problem for NSUPW, NLSW, and NLiW is PSPACE-COMPLETE.*

Proof. (In PSPACE). The quantitative universality problem for NSUPW, NLSW, and NLiW can be reduced in polynomial time to respectively the boolean universality problem for finite word languages, for infinite word Büchi languages, and for infinite word co-Büchi languages.

(PSPACE-HARD). The boolean universality problem for finite word languages, for infinite word Büchi languages, and for infinite word co-Büchi languages can be reduced in polynomial time to their quantitative counterparts by assigning weight 1 to the transitions entering an accepting state, and weight 0 to the other transitions, and taking the threshold $\nu = 1$. ■

Theorem 18. *The quantitative universality problem for deterministic DLAW and DDiW is decidable in polynomial time.*

Proof. It follows from the results of Theorem 5. ■

Theorem 19. *The quantitative language equivalence problem for NSUPW, NLSW, and NLiW is PSPACE-COMPLETE.*

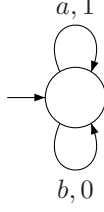


Fig. 8. A deterministic weighted automaton.

Proof. (In PSPACE). The results follow from Theorem 4.

(PSPACE-HARD). The boolean language equivalence problem for finite word languages, for infinite word Büchi languages, and for infinite word co-Büchi languages can be reduced in polynomial time to their quantitative counterparts by assigning weight 1 to the transitions entering an accepting state, and weight 0 to the other transitions. ■

Theorem 20. *The quantitative language equivalence problem for deterministic DLAW and DDtW is decidable in polynomial time.*

Proof. It follows from the results of Theorem 5. ■

Proof of Theorem 11. Consider the deterministic automaton A (shown in Fig. 8) that consists of a single self-loop state with weight 1 for a and 0 for b . For $j \geq 0$, consider the words $w_j = (b^j a)^\omega$ and $w'_j = a^j b^\omega$. Then we have $L_A(w_j) = \frac{1}{j+1}$ if A is interpreted as a DLAW, and $L_A(w'_j) = \frac{1-\lambda^j}{1-\lambda}$ if A is interpreted as a DDtW, *i.e.* the automaton A has infinitely many output values. The possible output value set for NBW, NCW, NLtW and NLsW is finite. Hence the result follows. ■

B Determinization of N when R_λ is finite

Lemma 9. *For $\frac{1}{2} < \lambda < 1$ and $R_\lambda = \{\frac{r_a(w)-r_b(w)}{\lambda^{|w|}} \mid w \in \Sigma^*\} \cap [-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}]$, the Disc-automaton N of Fig. 6 can be determinized if the set R_λ is finite.*

Proof sketch. We construct the DDtW $A_D = \langle Q, I, \Sigma, \delta, \mu \rangle$ such that $L_{A_D} = L_N$ as follows:

- $Q = R_\lambda \cup \{s_a, s_b\}$;
- $I = \{0\}$ (notice that $0 \in R_\lambda$ since $r_a(\epsilon) - r_b(\epsilon) = 0$);

- $\Sigma = \{a, b\}$;
- δ contains the following transitions:
 - $(s_a, a, s_a), (s_a, b, s_a), (s_b, a, s_b), (s_b, b, s_b)$
 - all the transitions (s, a, s') such that $s' = \begin{cases} s_a & \text{if } \frac{s+1}{\lambda} \geq \frac{1}{1-\lambda} \\ s_b & \text{if } \frac{s+1}{\lambda} \leq \frac{-1}{1-\lambda} \\ \frac{s+1}{\lambda} & \text{otherwise} \end{cases}$
 - all the transitions (s, b, s') such that $s' = \begin{cases} s_a & \text{if } \frac{s-1}{\lambda} \geq \frac{1}{1-\lambda} \\ s_b & \text{if } \frac{s-1}{\lambda} \leq \frac{-1}{1-\lambda} \\ \frac{s-1}{\lambda} & \text{otherwise} \end{cases}$
- $\mu : \delta \rightarrow \mathbb{Q}$ is defined as follows:
 - $\mu(s_a, a, s_a) = \mu(s_b, b, s_b) = 1$;
 - $\mu(s_a, b, s_a) = \mu(s_b, a, s_b) = 0$;
 - $\mu(s, a, s') = 1$ and $\mu(s, b, s') = 0$ for all $s, s' \in R_\lambda$
 - $\mu(s, a, s_a) = 1$ and $\mu(s, b, s_b) = 1 - s$ for all $s \in R_\lambda$

The correctness of this construction can be justified by the following observations:

1. If $s = \frac{r_a(w) - r_b(w)}{\lambda^{|w|}}$, then $\frac{r_a(w.a) - r_b(w.a)}{\lambda^{|w.a|}} = \frac{r_a(w) + \lambda^{|w|} - r_b(w)}{\lambda^{|w|+1}} = \frac{s+1}{\lambda}$ and similarly $\frac{r_a(w.b) - r_b(w.b)}{\lambda^{|w.b|}} = \frac{s-1}{\lambda}$
2. The automaton A_D gives weight $r_a(w)$ to the finite words w such that $\text{Post}(w) \in R_\lambda$. Assume that A_D is in state $s \in R_\lambda$ after reading a finite word w . Then, if $(s, a, s_a) \in \delta$, the correct value is given to every continuation of $w.a$, since $w.a$ uniformly prefers a (cf Lemma 5). If $(s, b, s_b) \in \delta$, then $w.b$ uniformly prefers b and the value given by A_D to $w.b$ is $r_a(w) + \lambda^{|w|} \cdot (1 - s) = r_a(w) + \lambda^{|w|} - (r_a(w) - r_b(w)) = r_b(w.b)$.
3. By Lemma 5, if an infinite word w has all its prefixes ambiguous, then $r_a(w) = r_b(w)$ and thus the value given by A_D to w is correct (it is $r_a(w)$).

■

Fig. 9 shows a DDIW defining the language L_N for $\lambda = \frac{\sqrt{5}-1}{2}$ (which gives a finite set R_λ).

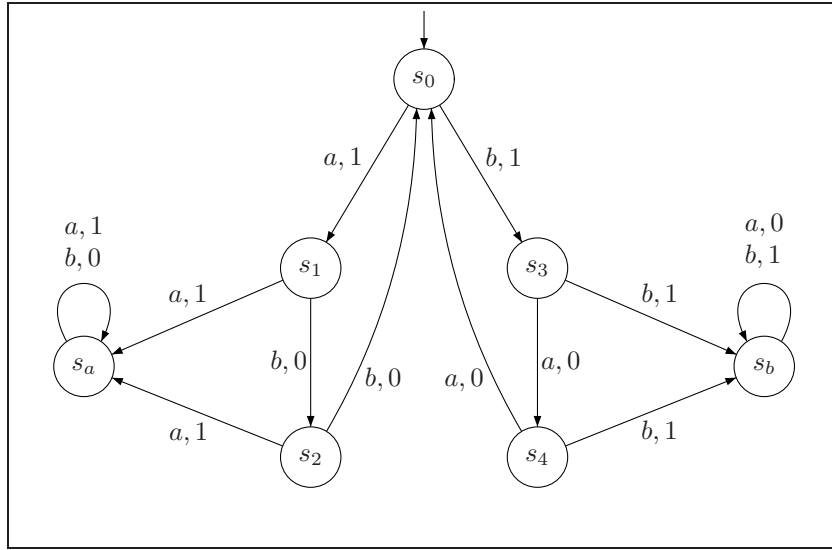


Fig. 9. A_D for $\lambda = \frac{\sqrt{5}-1}{2}$ (i.e. $1 = \lambda + \lambda^2$).